Additive Representation under Idempotent Attention: Online Appendix

Dayang Li*

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This online appendix contains five parts. First, Appendix A provides the characterization of the AR-IA without assuming positive utility functions. Second, in Appendix B, we discuss the uniqueness of utility functions under similarity transformation and the uniqueness of consistent pairs. Third, the practices of revealed attention under AR-IAs and AR-AFs are introduced using the weakest criteria in Appendix C. Fourth, we present several results regarding interpersonal comparisons of attention capacity in Appendix D. Finally, we introduce the Borda-IA as an application of the AR-IA in Appendix E

A AR-IAs without $u \geq 0$

The paper assumes that $u \geq 0$. As a result of this assumption, we can characterize alternatives with zero utility and state the IB axiom in a straightforward manner. Despite its absence, we are still able to characterize the AR-IA.

Definition. *$\succcurlyeq$ on $\mathcal{X}$ admits a* ***General Additive Representation under Idempotent Attention (General AR-IA)*** *if there exist an idempotent attention rule $\Gamma : \mathcal{X} \to \mathcal{X}$, and a function $u : X \to \mathbb{R}$ such that for any $S, T \in \mathcal{X}$,*

$$S \succcurlyeq T \iff \sum_{s \in \Gamma(S)} u(s) \geq \sum_{t \in \Gamma(T)} u(t).$$

Given any pair of above Γ and u, we say that $\succcurlyeq$ on $\mathcal{X}$ admits a General AR-IA under (Γ, u).

*Department of Economics, University of California Riverside. Email: dayang.li@email.ucr.edu.

Of course, we still need WO.

Axiom 1. *(WO: Weak Order)* $\succcurlyeq$ *on* $\mathcal{X}$ *is complete and transitive.*

Based on the proof of the characterization of the AR-IA, we can associate a set S to one of its corresponding basic sets to construct the idempotent attention rule. Moreover, the basic sets must catch the DM's full attention. Consequently, we only need to ensure that the system of inequalities induced by the $\succcurlyeq$ on $\mathcal{B}$ has a solution. The condition still follows Kraft et al. (1959), Scott (1964), Krantz et al. (1971) and Fishburn (1992), which is known as Finite Cancellation or Strong Additivity.

Axiom 2. *(FC: Finite Cancellation) There do not exist a positive integer* m *and* $\{S_n\}_{n=1}^m$, $\{T_n\}_{n=1}^m \subseteq \mathcal{B}$ *where* $S_n \succcurlyeq T_n$ *for all* n, *and* $S_n \succ T_n$ *for some* n, *such that* $\sum_{n=1}^m \mathbb{1}_{S_n}(x) = \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ *for all* $x \in X$ *where* $\mathbb{1}$ *is the indicator function.*

FC follows a similar logic to NR. According to the FC axiom, if two subcollections of basic sets contain exactly the same content, it cannot be the case that the DM prefers one over the other. Obviously, it imposes a consistent condition on aggregating preference on X.

Theorem 1. $\succcurlyeq$ *on* $\mathcal{X}$ *admits a General AR-IA if and only if it satisfies WO and FC.*

Proof. There is no doubt that WO is necessary. In order to demonstrate the necessity of FC, assume that $\succcurlyeq$ on $\mathcal{X}$ admits a General AR-IA under (Γ, u). By contradiction, assume that FC does not hold. Fix a positive integer m and $\{S_n\}_{n=1}^m$, $\{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \succcurlyeq T_n$ for all n, and $S_n \succ T_n$ for some n. We then know that $\sum_{n=1}^m \sum_{x\in S_n} u(x) > \sum_{n=1}^m \sum_{x\in T_n} u(x)$. However, $\sum_{n=1}^m \sum_{x\in S_n} u(x) = \sum_{n=1}^m \sum_{x\in T_n} u(x)$ when $\sum_{n=1}^m \mathbb{1}_{S_n}(x) = \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \in X$.

To ensure sufficiency, we can follow the proof of characterization of the AR-IA to construct the idempotent attention rule Γ with $\Gamma(\mathcal{X}) = \mathcal{B}$. FC guarantees the AR on $\mathcal{B}$, which follows directly from Lemma 0 in Kraft et al. (1959). □

B More Results about the Uniqueness of (Γ, u)

B.1 Uniqueness of Utility Functions

As we mentioned above, if u is a consistent utility function for the AR-IA (AR-AF), then cu with $c > 0$ is also a consistent utility function for the representation.[1] It is natural to

[1] This implies that $\{(\Gamma_i, u_i)\}_{i\in I_{\succcurlyeq}}$ and $\{(\Gamma_j, u_j)\}_{j\in J_{\succcurlyeq}}$ must contain infinitely many consistent pairs.

ask whether for any two consistent utility functions u and v for the AR-IA (AR-AF), there is a $c > 0$ such that $u = cv$. If it is true, then say that the utility function is unique up to similarity transformation (Fishburn (1992)).

Definition. *Assume that* $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-IAs. We say the utility function is* ***unique up to similarity transformation under the AR-IA*** *if for any* $(\Gamma_{i_1}, u), (\Gamma_{i_2}, v) \in \{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ *there is a* $c > 0$ *such that* $u = cv$*. Similarly, Assume that* $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-AFs. We say the utility function is* ***unique up to similarity transformation under the AR-AF*** *if for any* $(\Gamma_{j_1}, u), (\Gamma_{j_2}, v) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ *there is a* $c > 0$ *such that* $u = cv$*.*

For both AR-IAs and AR-AFs, the utility function is not necessarily unique up to similarity transformation.

Example 1. *Let* $X = xyz$*. The preference* $\succcurlyeq$ *on* $\mathcal{X}$ *is* $xyz \succ xy \sim x \succ yz \sim y \succ xz \sim z$*. It is clear that* $\mathcal{B} = \{xyz, x, y, z\}$*. We consider two pairs of attention rules and utility functions. Let* $u_1(x) = 3, u_1(y) = 2, u_1(z) = 1$*, and*

$$\Gamma_1(S) = \begin{cases} x & \text{if } S = xy, \\ y & \text{if } S = yz, \\ z & \text{if } S = xz, \\ S & \text{otherwise.} \end{cases}$$

Let $u_2(x) = 2, u_2(y) = 1, u_2(z) = 0$*, and*

$$\Gamma_2(S) = \begin{cases} x & \text{if } S = xy, \\ y & \text{if } S = yz, \\ xz & \text{if } S = xz, \\ S & \text{otherwise.} \end{cases}$$

Clearly, $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-IAs and AR-AFs, and* (Γ_1, u_1) *and* (Γ_2, u_2) *can both be consistent pairs. However,* $\succcurlyeq$ *on* $\mathcal{X}$ *cannot be represented by* (Γ_2, u_1) *under the AR-IA and the AR-AF.*

We also cannot find a positive real number c *such that* $u_1 = cu_2$*. If such a* c *exists, it requires that* $u_1(z) = cu_2(z)$ *which is impossible.*

Proposition 1. *If the utility function is unique up to similarity transformation under the AR-IA, then $N = \emptyset$.*

Proof. Assume that the utility function is unique up to similarity transformation under the AR-IA. By Claim 1, we know that $Rank(A) = |X/_\sim| - 1$. By contradiction, suppose that $N = \emptyset$. Then, there is a $u \in \mathcal{U}_\succcurlyeq$ such that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. As a result, $u(y) = 0$ for all $y \in X$. Notice that it suggests that $x \sim y$ for all $x, y \in \mathcal{X}$. Moreover, $\mathcal{B} = \{S \in \mathcal{X} : |S| = 1\}$. In this situation, we can let $v(x) = 1$ for all $x \in X$. It is clear that $v \in \mathcal{U}_\succcurlyeq$ while there is no $c > 0$ such that $u = cv$. □

Suppose that the utility function is unique up to similarity transformation. Fix any two consistent utility functions u, v with $u = cv$ where $c > 0$. For any arbitrary $x \in X$, we know that $u(x) = cv(x)$. In view of the fact that u and v are strictly positive utility functions, $c = \frac{u(x)}{v(x)}$. As a result, $u(y) = cv(y) = \frac{v(y)}{v(x)}u(x)$ for all $y \in X$. Hence, $u(x)$ and $u(y)$ have a proportional relationship for any $x, y \in X$.

Condition 1. *(Proportion) For any $x, y \in X$ with $x \succcurlyeq y$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $0 < \sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x) \leq \sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$.*

The Condition 1 ensures that there is a proportional relationship between the utility of any two alternatives. It turns out to be the necessary and sufficient condition for the utility function to be unique up to similarity transformation under the AR-IA. Furthermore, this condition is also valid for the case of AR-AFs.

Theorem 2. *Assume that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs (AR-AFs). The utility function is unique up to similarity transformation if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies Proportion.*

Proof. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Let $\mathcal{U}_\succcurlyeq$ be the collection of consistent utility functions.

We begin with proving the if part. Consider the equivalent class $[x] = \{y \in X : x \sim y\}$, we know we can partition X by the equivalent classes. Now, let $X/_\sim = \{x_i\}_{i=1}^I$ where $x_i \succ x_j$ for all $i < j$. Suppose that for any $x_i, x_j \in X$ with $x_i \succcurlyeq x_j$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $\sum_{n=1}^m \mathbb{1}_{S_n}(x_i) \geq \sum_{n=1}^m \mathbb{1}_{T_n}(x_j)$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$. Since $\succcurlyeq$ on $\mathcal{X}$ admits the AR-IA under (Γ, u), we know that there is a pair of real numbers c_i, c_j such that

$c_i u(x_i) = c_j u(x_j)$ where $c_i > c_j$. By induction, we have $c_1 u(x_1) = c_2 u(x_2) = \cdots = c_I u(x_I)$ where $c_1 > c_2 > \cdots > c_I$. Furthermore, for any other AR-IA pair (Γ', v), we still have $c_1 v(x_1) = c_2 v(x_2) = \cdots = c_I v(x_I)$ where $c_1 > c_2 > \cdots > c_I$. Hence, $u = \frac{u(x_1)}{v(x_1)} v$.

For the converse direction, suppose that $\mathcal{U}_{\succcurlyeq}$ is up to under similarity transformation. We first know that if $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA under (Γ, u), then (Γ, cu) also be a consistent AR-IA pair for any $c > 0$. Consequentially, if $\mathcal{U}_{\succcurlyeq} \neq \emptyset$, then $|\mathcal{U}_{\succcurlyeq}| = \infty$.

Let's consider the linear system induced by $\sim$ on $\mathcal{B}$. If $B \sim B'$, we know $\sum_{b \in B} u(b) - \sum_{b'} u(b') = 0$. the system then can be written as $Au(X) = 0$ where A is the coefficient matrix and $u(X) = (u(x_1), ..., u(x_I))^T$.

Claim 1. *If the utility function is unique up to similarity transformation, then* $Rank(A) = |X/_{\sim}| - 1$.

Proof. By contradiction, suppose that the linear system induced by $\sim \cap (\mathcal{B} \times \mathcal{B})$ does not have $|X/_{\sim}| - 1$ linearly independent equations. We know the linearly independent equations must be smaller than $|X/_{\sim}| - 1$, otherwise $u(x) = 0$ for all x. In this case, $u(x) = 1$ for all x still represents $\succcurlyeq$ while the utility is not unique under similarity transformation. Now, suppose that $Rank(A) = |X/_{\sim}| - k$ where $k > 1$. The solution to $Au(X) = 0$ then has k parameters. In other words, fix any $(u(x_1), .., u(x_k)) = (a_1, ..., a_k)$, $u(x_{k+i})$ should be a linear combination of $(a_1, ..., a_k)$ for any $i \leq I - k$. Suppose that $(u(x_1), .., u(x_k)) = (1, ..., k)$ and $(v(x_1), .., v(x_k)) = (3, ..., k+3)$, we know there is no $c > 0$ such that $u = cv$. □

By Gaussian elimination, if $Rank(A) = |X/_{\sim}| - 1$, then the last nonzero row of A can be reduced as $(0, ..., 0, c_i, c_j)$ which implies that $c_i u(x_i) = c_j u(x_j)$ for arbitrary x_i, x_j. Moreover, if $x_i \succcurlyeq x_j$, then $c_i \leq c_j$. Thus, for any $x, y \in X$ with $x \succcurlyeq y$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $0 < \sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x) \leq \sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$. When it comes to the uniqueness of (Γ, u), we can combine the uniqueness of utility functions and the uniqueness of attention rules. □

The proportional relationship between the utility of alternatives must be determined by the indifference relationship on $\mathcal{B}$. Intuitively, the linear system induced by $\sim \cap (\mathcal{B} \times \mathcal{B})$ must have one parameter. In other words, for the $x \succ y$ in the statement of Theorem 2, if $u(x)$ is fixed as some positive number, then $u(y)$ must be $cu(x)$ where $c = \frac{\sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x)}{\sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)}$.

Example 2. *Let $X = xyz$, the DM's preference on $\mathcal{X}$ is $xyz \sim z \sim xy \succ x \sim xz \sim yz \sim y$. Notice that $\mathcal{B} = \{xy, x, y, z\}$. We want to show that $u(z) = 2u(x)$ for all $u \in \mathcal{U}$. Consider, $\{xy, x\}\{z, y\}$, we know that $xy \sim z$ and $x \sim y$. Both collections contain one y. The $\{xy, x\}$ collection contains two x, and no z. The latter collection contains one z and no x. Clearly, if $u(x)$ is fixed, then $u(z) = 2u(x)$. Furthermore, given another $v \in \mathcal{U}_{\succcurlyeq}$, we also have $v(z) = 2v(x) = 2v(y)$. Let $c = \frac{v(x)}{u(x)}$, we then know that $v = cu$. The utility function is unique up to similarity transformation.*

B.2 Uniqueness of Consistent Pairs

Definition. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA (AR-AF). The AR-IA (AR-AF) representation is unique if the attention rule is unique and the utility function is unique up to similarity transformation.*

In light of the requirement that the utility function be unique, it is evident that there are no null alternatives. $N = \emptyset$, as the requirement of the uniqueness of attention rules, is redundant for characterizing the uniqueness of (Γ, u).

Corollary 3. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA. The AR-IA representation is unique if and only if*

(i) $\succcurlyeq$ on $\mathcal{X}$ satisfies the Proportion condition;

(ii) for every $S \in \mathcal{X}$ there is a unique $B \in \mathcal{B}$ such that $B \subseteq S$ and $B \sim S$, i.e., $|\mathcal{B}(S)| = 1$ for all $S \in \mathcal{X}$.

Corollary 4. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA. The AR-IA representation is unique if and only if*

(i) $\succcurlyeq$ on $\mathcal{X}$ satisfies the Proportion condition;

(ii) for any S there is a unique basic set B such that $T \sim S \sim B$ for all T with $B \subseteq T \subseteq S$.

C Revealed Attention

The paper analyzes the revealed attention by imposing stringent criteria for both AR-IAs and AR-AFs. It is also possible to investigate them by imposing the weakest criteria. The combination of these two practices gives us a comprehensive picture of how DMs form their attention.

C.1 Revealed Attention under AR-IAs

Suppose that the $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs, and let $\{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ be the collection of all consistent pairs of idempotent attention rule and utility function. When one of the consistent idempotent attention rules suggests that the DM will pay attention to s in S, an alternative s might catch the DM's attention. Formally,

Definition. *An alternative s **weakly catches attention under AR-IAs** in S if $s \in \Gamma_i(S)$ for some $i \in I_{\succcurlyeq}$.*

Let $\bar{\mathcal{B}}(S) := \{T \subseteq S : \exists B \in \mathcal{B}(S) \text{ s.t. } T \sim B \text{ and } T \setminus B \subseteq N\}$. We know that every $T \in \bar{\mathcal{B}}(S)$ can be the DM's attention on S. Accordingly, if $s \in T$ for some $T \in \bar{\mathcal{B}}(S)$, then s should weakly catch the attention of AR-IAs.

Corollary 5. *An alternative s weakly catches attention under AR-IAs in S if and only if there is a $T \in \bar{\mathcal{B}}(S)$ such that $s \in T$.*

Proof. As we know, if $T \in \bar{\mathcal{B}}(S)$, then there is a Γ_i with $\Gamma_i(S) = T$ for some $i \in I_{\succcurlyeq}$. The Corollary 5 is a direct result of this observation. □

In a similar manner, we can form the weakest criterion for a menu S that catches attention.

Definition. *A menu S **weakly catches full attention under AR-IAs** if $\Gamma_i(S) = S$ for some $i \in I_{\succcurlyeq}$.*

Under AR-IAs, the DM considers all alternatives in any basic set. The DM may pay attention to all alternatives in S if a set S is composited with one of its corresponding basic sets and some null alternatives.

Proposition 2. *A menu S weakly catches full attention under AR-IAs if and only if there is a corresponding basic set B of S such that $S \setminus B \subseteq N$.*

Proof. Suppose that a menu S weakly catches full attention under AR-IAs, and the consistent idempotent attention and utility function are Γ and u, respectively. If S is basic, we are done. If S is nonbasic, then for any $B \in \mathcal{B}(S)$, we have

$$\sum_{s \in \Gamma(S)} u(s) - \sum_{b \in B} u(b) = \sum_{s \in S} u(s) - \sum_{b \in B} u(b) = \sum_{s \in S \setminus B} u(s) = 0.$$

That is, $S \setminus B \subseteq N$. For the converse direction, it is clear that $S \setminus B = N$ for some $B \in \mathcal{B}(S)$ implies that $S = B \cup O$ for some $B \in \mathcal{B}(S)$ and $O \subseteq S \cap N$. Hence, there is an idempotent attention Γ that consistents with AR-IAs such that $\Gamma(S) = S$. $\square$

C.2 Revealed Attention under AR-AFs

We now assume that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs, and the collection of consistent pairs of attention filer and utility function is denoted as $\{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$. We can define an alternative that weakly catches attention under AR-AFs in the same manner as AR-AFs.

Definition. *An alternative s* ***weakly catches attention under AR-AFs*** *in S if $s \in \Gamma_j(S)$ for some $j \in J_{\succcurlyeq}$.*

We can construct a consistent attention filter by associating S to a $T \in \mathcal{IB}(S)$.

Proposition 3. *An alternative s weakly catches attention in S under AR-AFs if and only if there is a $T \in \mathcal{IB}(S)$ such that $s \in T$.*

Proof. We know that for any $T \in \mathcal{IB}(S)$, there is an attention filter that is consistent with AR-AFs such that $\Gamma(S) = T$. Take any attention filter Γ that is consistent with AR-AFs and a set S such that $\Gamma(S) \notin \mathcal{IB}(S)$, we then know that there is a Y where $\Gamma(S) \subseteq Y \subseteq S$ such that Y is not indifferent to S, which suggests that Γ is not an attention filter. As a result, the theorem is relatively straightforward. $\square$

The weakest criterion of revealed attention can also be applied to sets under AR-AFs.

Definition. *A menu S* ***weakly catches full attention under AR-AFs*** *if $\Gamma_j(S) = S$ for some $j \in J_{\succcurlyeq}$.*

We know that any $T \in \mathcal{IB}(S)$ may serve as $\Gamma_j(S)$ for some $j \in J_{\succcurlyeq}$. As a result, if $S \in \mathcal{IB}(S)$, then S should weakly attract the full attention of the DM under AR-AFs. This property can be further investigated by observing that the set difference between S and any of its corresponding basic sets is a subset of N. As a result, this characterization is identical to that found in AR-IAs.

Proposition 4. *A menu S weakly catches full attention under AR-AFs if and only if there is a corresponding basic set B of S such that $S \setminus B \subseteq N$.*

Proof. Suppose that a menu S weakly catches full attention under AR-AFs, and (Γ_j, u_j) the consistent pair of AR-AF. We then know $\Gamma_j(S) = S$. If S is basic, then $S \setminus S = \emptyset \subseteq N$. If S is not basic, then there is a corresponding basic set B of S such that $\sum_{s \in S \setminus B} u_j(s) = 0$, which implies that $S \setminus B \subseteq N$. For the converse direction, taking a $B \in \mathcal{B}(S)$ such that $S \setminus B \subseteq N$. Then, we know that $S \in \mathcal{IB}(S)$, which suggests that there is an attention filer Γ'_j that is consistent with AR-AFs such that $\Gamma'_j(S) = S$. □

D Relative Attention

One of the reasons for DMs' limited attention is their limited cognitive capacity. DMs' cognitive abilities can be inferred from basic sets when we interpret basic sets as the images of their attention rule.

In order to compare interpersonal attention, we need to know which alternatives attract the DMs' attention for each menu. When their attention rules are idempotent, it is equivalent to finding the relationship between the corresponding basic sets of each menu. Intuitively, when DM 1 is more attentive than DM 2, then, given any menu, an alternative that attracts DM 2's attention should catch DM 1's attention as well. Occasionally, we are unable to pin down the attention of the DMs. We then use the most stringent condition for this comparison.

Suppose that two DMs' preference $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{X}$ both admit AR-IAs, and denote $\{(\Gamma_{i_1}, u_{i_1})\}_{i_1 \in I_{\succcurlyeq_1}}$ and $\{(\Gamma_{i_2}, u_{i_2})\}_{i_2 \in I_{\succcurlyeq_2}}$ as the collection of the consistent pairs of AR-IA for DM 1 and DM 2, respectively.

Definition. *Suppose that $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{X}$ admit AR-IAs. $\succcurlyeq_1$ is more attentive than $\succcurlyeq_2$ if given any $i_1 \in I_{\succcurlyeq_1}$ and $S \in \mathcal{X}$, $\Gamma_{i_2}(S) \subseteq \Gamma_{i_1}(S)$ for all $i_2 \in I_{\succcurlyeq_2}$.*

Under AR-AFs, relative attention can be defined similarly. Let's denote the consistent pairs of AR-AFs as $\{(\Gamma_{j_1}, u_{j_1})\}_{j_1 \in J_{\succcurlyeq_1}}$ and $\{(\Gamma_{j_2}, u_{j_2})\}_{j_2 \in J_{\succcurlyeq_2}}$ for DM 1 and DM 2 respectively.

Definition. *Suppose that $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{X}$ admit AR-AFs. $\succcurlyeq_1$ is more attentive than $\succcurlyeq_2$ if given any $j_1 \in J_{\succcurlyeq_1}$ and $S \in \mathcal{X}$, $\Gamma_{j_2}(S) \subseteq \Gamma_{j_1}(S)$ for all $j_2 \in J_{\succcurlyeq_2}$.*

By $\mathcal{B}_{\succcurlyeq_1}$ and $\mathcal{B}_{\succcurlyeq_2}$ we respectively denote the collection of basic sets of $\succcurlyeq_1$ and $\succcurlyeq_2$. Since $\mathcal{B}_{\succcurlyeq_1} \subseteq \Gamma_{i_1}(\mathcal{X})$ for all $i_1 \in I_{\succcurlyeq_1}$, we can focus on the $\mathcal{B}_{\succcurlyeq_1}$. Similarly, we denote $\bar{\mathcal{B}}_{\succcurlyeq_1}(S)$ and $\bar{\mathcal{B}}_{\succcurlyeq_2}(S)$ as $\bar{\mathcal{B}}(S)$ for $\succcurlyeq_1$ and $\succcurlyeq_2$ respectively. If, given any menu S, an alternative $s \in T$

for some $T \in \bar{\mathcal{B}}_{\succsim_2}(S)$, then s must be included in all the corresponding basic sets of S in under $\succsim_1$.

Corollary 6. *Suppose that $\succsim_1$ and $\succsim_2$ on $\mathcal{X}$ admit AR-IAs. $\succsim_1$ is more attentive than $\succsim_2$ if and only if, for any $S \in \mathcal{X}$, $\bigcup_{T' \in \bar{\mathcal{B}}_{\succsim_2}(S)} T' \subseteq B$ for all $B \in \mathcal{B}_{\succsim_1}(S)$.*

Proof. By Corollary 5, $\succsim_1$ is more attentive than $\succsim_2$ if and only if, for any $S \in \mathcal{X}$, $\bigcup_{T' \in \bar{\mathcal{B}}_{\succsim_2}(S)} T' \subseteq T$ for all $T \in \bar{\mathcal{B}}_{\succsim_1}(S)$. Since $\mathcal{B}_{\succsim_1}(S) \subseteq \bar{\mathcal{B}}_{\succsim_1}(S)$ for any S, $\bigcup_{T' \in \bar{\mathcal{B}}_{\succsim_2}(S)} T' \subseteq T$ for all $T \in \bar{\mathcal{B}}_{\succsim_1}(S)$ if and only if $\bigcup_{T' \in \bar{\mathcal{B}}_{\succsim_2}(S)} T' \subseteq B$ for all $B \in \mathcal{B}_{\succsim_1}(S)$. □

When DMs' attention rules are attention filters, given any menu S, their attention to S is an element in $\mathcal{IB}(S)$. Let $\mathcal{IB}_{\succsim_1}(S)$ and $\mathcal{IB}_{\succsim_2}(S)$ denote the $\mathcal{IB}(S)$ for DM 1 and DM 2 respectively. In this case, the relative attention can be understood as follows: If an alternative x is included in T' where $T' \in \mathcal{IB}_{\succsim_2}(S)$, then x should also be included in T for all $T \in \mathcal{IB}_{\succsim_1}(S)$.

Corollary 7. *Suppose that $\succsim_1$ and $\succsim_2$ on $\mathcal{X}$ admit AR-AFs. $\succsim_1$ is more attentive than $\succsim_2$ if and only if, for any $S \in \mathcal{X}$, $\bigcup_{T' \in \mathcal{IB}_{\succsim_2}(S)} T' \subseteq T$ for all $T \in \mathcal{IB}_{\succsim_1}(S)$.*

Proof. Since $\Gamma_{j_1}(S) = T$ for some $T \in \mathcal{IB}_{\succsim_1}(S)$ and $\Gamma_{j_2}(S) = T'$ for some $T' \in \mathcal{IB}_{\succsim_2}(S)$, the statement holds. □

The definition of relative attention allows for the heterogeneity of alternatives in the formation of attention. When some alternatives are objectively easier to catch the attention of DMs than others, relative attention can be considered as an alternative to alternative comparison. When the attention-forming procedures are homogeneous between alternatives, each alternative has the same level of difficulty in attracting DMs' attention. Similar interpretations can be found in Geng and Özbay (2021) and Geng (2022). In this scenario, we only need to consider the cardinalities of the sets in $\bar{\mathcal{B}}(S)$ and $\mathcal{IB}(S)$ for the AR-IA and AR-AF, respectively.

E An Application of the AR-IA: Borda-IA

According to the AR-IA, the DM aggregates utility to form preferences over menus. Depending on the individual, the utility function in the AR-IA varies. However, utility functions are often specified in practice, such as the Borda score. Suppose that $|X/_{\sim}| = k$, by using the Borda score, the DM assigns $k - i + 1$ for the i-th preferred alternatives in X

(see, e.g., Baigent and Xu (2004) and Darmann and Klamler (2019)).[2] We may define the Borda score as a function $b : X \to \mathbb{N}$. In cases where alternatives are mutually compatible in each menu, the DM is able to add the Borda scores of the alternatives in each menu to form $\succcurlyeq$. [3]

If the DM is using the Borda score to evaluate menus and has an idempotent attention rule, then $\succcurlyeq$ should admit the AR-IA under (Γ, b).

Definition. *$\succcurlyeq$ on $\mathcal{X}$ admits a* ***Borda score under idempotent attention (Borda-IA)*** *if there is an idempotent attention rule $\Gamma : \mathcal{X} \to \mathcal{X}$ such that*

$$S \succcurlyeq T \iff \sum_{s \in \Gamma(S)} b(s) \geq \sum_{t \in \Gamma(T)} b(t)$$

where $b : X \to \mathbb{N}$ is the Borda score.

A Borda-IA is an AR-IA in which the Borda score b is used as the utility function. A partition of X can be formed by the inverse image of b. When $|X/_{\sim}| = k$, $b^{-1}(k)$ denotes the most preferred alternatives. Similarly, $b^{-1}(1)$ represents the collection of least preferred alternatives, i.e., $MIN(X, \succcurlyeq)$. For simplicity, we denote $b^{-1}(0) = x_0 = \emptyset$. The Borda score implies that $x_i \in b^{-1}(i)$ should be indifferent to $x_{i-1}x_1$ with $x_{i-1} \in b^{-1}(i-1)$ and $i \geq 1$ if the DM pays full attention to both x_i and $x_{i-1}x_1$. Therefore, for some $S \in \mathcal{X}$, suppose that $x_i, x_j \in S$ and $x_{i+1}, x_{j-1} \notin S$. When the DM pays full attention, the preference should exhibit $S \sim (S \setminus x_i x_j) \cup x_{i+1} \cup x_{j-1}$.

Definition. *For any $S \in \mathcal{X}$, $x_i \in b^{-1}(i) \cap S$, $x_j \in b^{-1}(j) \cap S$, $x_{i+1} \in b^{-1}(i+1) \cap S^c$, and $x_{j-1} \subseteq b^{-1}(j-1) \cap S^c$ where $i < k$ and $j \geq 1$, $(S \setminus x_i x_j) \cup (x_{i+1} \cup x_{j-1})$ is a* ***basic equivalent transformation*** *of S. Let us denote it as $S \cong (S \setminus x_i x_j) \cup (x_{i+1} \cup x_{j-1})$. A menu T is an* ***equivalent transformation*** *of S if $S \; Tran(\cong) \; T$ where $Tran(\cong)$ is the transitive closure of $\cong$. Let us denote it as $\simeq$.*

Through the equivalent transformation of S, we can transform it into a menu T with the same Borda score. Instead of limited attention, this transformation is determined by

[2] Another Borda score is used in previous literature, where the DM assigns $k - i$ for the i-th preferred alternative. This situation results in $k - 1$ being given to the most preferred alternative, and 0 being given to the least preferred alternative. For simplicity, we will focus on the Borda score in which all alternatives possess strict positive utility.

[3] Alcantud and Arlegi (2008) interprets b as a "categorizing" function. Specifically, the DM partitions the alternative set X based on a number of characteristics. The DM has a linear preference for the collection of characteristics represented by the function b. As a result, the value of a menu is determined by adding the value of its characteristics together.

the Borda score. In light of this, even though the DM is limitedly attentive, $\succcurlyeq$ on $\Gamma(\mathcal{X})$ should satisfy the monotonicity axiom.

Axiom 3. *(EM: Extended Monotonocity) For any $S, T \in \mathcal{X}$ and $B_1, B_2 \in \mathcal{B}$, if $B_1 \simeq S \supseteq T \simeq B_2$, then $B_1 \succcurlyeq B_2$.*

The EM axiom posits an extra additive requirement for the AR-IA. For two basic sets B_1 and B_2, if an equivalent transformation of B_1 is a superset of an equivalent transformation of B_2, then the DM should prefer B_1 to B_2. It turns out that the requirement of the Borda-IA can be characterized by the EM axiom.

Theorem 8. *$\succcurlyeq$ on $\mathcal{X}$ admits a Borda-IA if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies EM.*

Proof. We first notice that $\simeq$ is reflexive. Let us first permutate X by using b, and denote this permutation by π. To be specific,

$$X = \left\{x_{m_1}, ..., x_{m_{|b^{-1}(m)|}}, ..., x_{1_1}, ..., x_{1_{|b^{-1}(1)|}}\right\} = \left\{x_{\pi_1}, ..., x_{\pi_{|b^{-1}(m)|}}, ..., x_{\pi_{|X|}}\right\}$$

where $x_{i_\alpha} \in b^{-1}(i)$ for all $1 \leq \alpha \leq |b^{-1}(i)|$. We then can consider X a finite sequence with permutation π, i.e., $X = X_\pi$.

Claim 2. *For any $S \in \mathcal{X}$, there is a $T \in \mathcal{X}$ with $S \simeq T$ such that either $|T| = 1$ or the first $|T| - 1$ elements are $\{x_{\pi_i}\}_{i=1}^{|T|-1}$ when $|T| \geq 2$.*

Proof. Take any $S \in \mathcal{X}$, assume that S is a subsequence of X_π. Suppose that S starts being different from X_π at some n-th element, and list the remaining part of S as $\{s_n, ..., s_{n+k}\}$. If S is identical to the first $|S|$ or $|S| - 1$ elements of X_π, then we are done. We, therefore, assume that $k \geq 1$. By the equivalent transformation, we also can assume that the DM strictly prefers $x_{\pi_{|s|-k}}$ over s_n. This suggests that $s_n \notin b^{-1}(m)$. Suppose that $s_n \in b^{-1}(i)$ where $i < m$ and $s_{n+k} \in b^{-1}(j)$ where $j \geq 1$. By the equivalent transformation, we know that $S \simeq S \setminus s_n s_{n+k} \cup x_{i+1} \cup x_{j-1}$ where $x_{i+1} \in b^{-1}(i+1)$ and $x_{j-1} \in b^{-1}(j-1)$. We can assume that $x_{i+1} = x_{\pi_\beta}$ where β is the smallest number of all $x_{\pi_\alpha} \in b^{-1}(i+1)$. We can assume the same thing for x_{j-1}, and denote $S \setminus s_n s_{n+k} \cup x_{i+1} \cup x_{j-1}$ as S^1. If S^1 satisfies the requirement for the claim, we are done. If not, we can repeat the previous steps until the requirement is fulfilled. □

Furthermore, for any $S \in \mathcal{X}$, let T be the equivalent transformation that meets the condition, and suppose that the $|T|$-th element is different from $x_{\pi_{|T|}}$. We can make one more step of the equivalent transformation by giving the lowest available index for the $|T|$-th elements in its equivalent class. If T coincides with the first $|T|$-th element of X_π, we can leave it as T. In this way, we can transform any S into a unique T^1. Let's denote this T^1 as S_π, i.e., $S \simeq S_\pi$.

Corollary 9. *For any $S, T \in \mathcal{X}$, $S \simeq T$ if and only if $\sum_{s\in S} b(s) = \sum_{t\in T} b(t)$.*

Proof. By the previous Claim, we know that there is a S_π and T_π such that $S \simeq S_\pi$ and $T \simeq T_\pi$. Moreover, $\sum_{s\in S_\pi} b(s) = \sum_{t\in T_\pi} b(t)$. Since $S_\pi = T_\pi$, we know that $\sum_{s\in S} b(s) = \sum_{t\in T} b(t)$. For the converse direction, take any $S \in \mathcal{X}$, we know that $S \simeq S_\pi$. Hence, $\sum_{s\in S} b(s) = \sum_{s\in S_\pi} b(s)$. Similarly, there is a T_π for T. Since $\sum_{s\in S} b(S) = \sum_{s\in S_\pi} b(s)$, $\sum_{s\in S_\pi} b(s) = \sum_{t\in T_\pi} b(t)$. Hence, $S_\pi = T_\pi$ which implies that $S \simeq T$. □

Now, we can prove the Theorem 8. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits a Borda-IA. Clearly, $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA. Moreover, take any $B_1, B_2 \in \mathcal{B}$, and $S, T \in \mathcal{X}$ such that $B_1 \simeq S \supseteq T \simeq B_2$. We then have that $\sum_{x\in\Gamma(B_1)b(x)} = \sum_{x\in B_1} b(x) = \sum_{s\in S} b(s)$. Similar equality also holds between B and T. Since $S \supseteq T$, $\sum_{s\in S} b(s) \geq \sum_{t\in T} b(t)$. As a result $\sum_{x\in\Gamma(B_1)} b(x) \geq \sum_{y\in\Gamma(B_2)} b(y)$ which implies that $B_1 \succcurlyeq B_2$.

For the converse direction, suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA and satisfies EM. We know that there is an idempotent attention Γ that is consistent with the AR-IAs such that $\Gamma(\mathcal{X}) = \mathcal{B}$. We then can focus on this Γ. By contradiction, suppose that (Γ, b) is not a consistent pair of the Borda-IA, i.e., there is a pair of $B_1, B_2 \in \mathcal{B}$ such that we cannot have $B_1 \succcurlyeq B_2 \iff \sum_{x\in\Gamma(B_1)} b(x) \geq \sum_{y\in\Gamma(B_2)} b(y)$. That is, $\sum_{x\in\Gamma(B_1)} b(x) \geq \sum_{y\in\Gamma(B_2)} b(y) \implies B_2 \succ B_1$. We first know that there are $B_{1\pi}, B_{2\pi}$ with $B_1 \simeq B_{1\pi}$ and $B_2 \simeq B_{2\pi}$ such that $\left|B_{j\pi} \setminus \bigcup_{i=1}^{|B_{j\pi}|} x_{\pi_i}\right| \leq 1$ for $j = 1, 2$. We can assume that $B_{1\pi} \neq B_{2\pi}$ Since $\sum_{t\in B_{1\pi}} b(t) \geq \sum_{t\in B_{2\pi}} b(t)$, we know that $|B_{1\pi}| \geq |B_{2\pi}|$. We need to consider the following cases:

Case 1: $\left|B_{1\pi} \setminus \bigcup_{i=1}^{|B_{1\pi}|} x_{\pi_1}\right| = 1$ and $\left|B_{2\pi} \setminus \bigcup_{i=1}^{|B_{2\pi}|} x_{\pi_1}\right| = 1$. If $MIN(B_{1\pi}, \succcurlyeq) = x_i \succ MIN(B_{2\pi}, \succcurlyeq) = x_j$, then $B_{1\pi} \setminus x_i \cup x_{i-j} x_j \simeq B_{1\pi} \supseteq B_{2\pi}$. Hence, $B_2 \succ B_1$ violates EM. If $MIN(B_{2\pi}, \succcurlyeq) = x_j \succ MIN(B_{1\pi}, \succcurlyeq) = x_i$, then we have $|B_{1\pi}| - |B_{2\pi}| \geq 1$. Moreover, $B_{1\pi} \setminus x_{\pi_{|B_{1\pi}|-1}} \cap x_j x_{j'} \simeq B_{1\pi} \supseteq B_{2\pi}$ where $x_{j'} \in b^{-1}(j - b(x_{\pi_{|B_{1\pi}|-1}}))$. Clearly, it also violates EM.

Case 2: $\left|B_{1\pi} \setminus \bigcup_{i=1}^{|B_{1\pi}|} x_{\pi_1}\right| = 0$ and $\left|B_{2\pi} \setminus \bigcup_{i=1}^{|B_{2\pi}|} x_{\pi_1}\right| = 0$. We then know that $MIN(B_{1\pi}, \succcurlyeq) = x_i \succ MIN(B_{2\pi}, \succcurlyeq) = x_j$. Following the same procedure, we can check that this violates EM.

Case 3: $\left|B_{1\pi} \setminus \bigcup_{i=1}^{|B_{1\pi}|} x_{\pi_1}\right| = 1$ and $\left|B_{2\pi} \setminus \bigcup_{i=1}^{|B_{2\pi}|} x_{\pi_1}\right| = 0$. We then have $B_{1\pi} \supseteq B_{2\pi}$. This case also violates EM.

Case 4: $\left|B_{1\pi} \setminus \bigcup_{i=1}^{|B_{1\pi}|} x_{\pi_1}\right| = 0$ and $\left|B_{2\pi} \setminus \bigcup_{i=1}^{|B_{2\pi}|} x_{\pi_1}\right| = 1$.We then have $B_{1\pi} \supseteq B_{2\pi}$. This case also violates EM. □

As a special evaluation rule, the Borda score indicates that there is a unit increment in utility between x_{i-1} and x_i. In the characterization of the Borda-IA, the condition for $b \in \mathcal{U}_{\succcurlyeq}$ is presented. Further, given the DM's preference for $\mathcal{X}$, when is it appropriate to say that the DM must use the Borda score in evaluating each menu?

Definition. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits a Borda-IA. The DM is* ***revealed to use Borda score*** *if for all $u \in \mathcal{U}_{\succcurlyeq}$, and $x, y \in X$, $\frac{u(x)}{b(x)} = \frac{u(y)}{b(y)}$.*

If the DM must use the Borda score to evaluate menus, then any consistent utility function should be produced by multiplying a positive number by b. It should be related to the uniqueness of the similarity transformation for the utility function.

Theorem 10. *The DM is revealed to use the Borda score if and only if for any $x \in b^{-1}(i), y \in MIN(X, \succcurlyeq)$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $\frac{\sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x)}{\sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)} = i$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$.*

Proof. Assume that $\succcurlyeq$ on $\mathcal{X}$ admits a Borda-IA, and the DM is revealed to use the Borda score. Take any $u, v \in \mathcal{U}_{\succcurlyeq}$ and $x, y \in X$, we know that $\frac{u(x)}{u(y)} = \frac{b(x)}{b(y)} = \frac{v(x)}{v(y)}$. It suggests that $u(x) = \frac{u(y)}{v(y)} v(x)$. Hence, the utility function is unique under similarity transformation. By the proof of Theorem 2, we can show this direction.

For the converse direction, we can assume that for any $x \in b^{-1}(k), y \in MIN(X, \succcurlyeq)$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $\frac{\sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x)}{\sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)} = k$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$. Since $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs, for any $u \in \mathcal{U}_{\succcurlyeq}$, we have $\frac{u(x_i)}{u(x_1)} = i$, as a result, $\frac{u(x_i)}{u(x_1)} = \frac{b(x_i)}{b(x_1)} = i$. It suggests that $\frac{u(x)}{b(x)} = \frac{u(y)}{b(y)}$ for all $x, y \in \mathcal{X}$. □

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