Additive Representation under Idempotent Attention*

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Abstract

This paper explores a scenario where a decision maker evaluates menus by adding up the utility of the options that attract her attention. We introduce a novel attention rule called the "idempotent attention rule" and examine additive representations under this rule. By utilizing idempotent attention rules, we are able to narrow our focus to a subset of menus to reveal both attention rules and utility functions. As a generalization of attention filters, this rule sheds light on how alternatives interact in forming attention.

JEL Codes: D01; D63; D91

Keywords: Additive representation; Limited attention; Idempotent attention; Attention filter

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1 Introduction

In our daily lives, a decision maker (DM) often needs to choose from various subscription-based services, such as online streaming, online gaming, or personal styling services. These services often offer menus with two main characteristics: (i) alternatives in each menu are mutually compatible (see Barberà et al. (2004)) for the DM can enjoy multiple items within a menu; (ii) the sheer number of items on each menu can make it difficult for the DM to pay attention to all of them. These challenges render the indirect utility representation unsuitable for the menu evaluation process, as it is silent on accounting for the utility of a menu that depends on multiple items and limited attention.

For evaluating menus with mutually compatible alternatives, Fishburn (1992) introduces the additive representation (AR).[1] Specially, a DM has a real-valued utility function u defined on a finite set of alternatives X. For any two nonempty subsets S and T of X, the DM prefers S to T if and only if $\sum_{s\in S} u(s) \geq \sum_{t\in T} u(t)$.

The AR captures the idea that the value of a menu depends on the multiple items in it. However, since it assumes that DMs are fully attentive, it is not suitable for the cases we discussed previously.

DMs are often limitedly attentive, meaning that they only pay attention to some of the alternatives, as suggested by many theoretical (cf. Masatlioglu et al. (2012); Manzini and Mariotti (2014); Lleras et al. (2017)) and empirical (cf. Goeree (2008); Wilson (2008)) studies. As a result, preferences over menus can be "distorted" due to limited attention. For instance, if the DM overlooks a "good" alternative in S, her evaluation of S will be lower than it should be. As a result, we require an appropriate representation theory that can guide menu evaluation by the limitedly attentive DM.

This paper examines additive representations under limited attention, with a particular focus on idempotent attention rules, namely, additive representations under idempotent attention (AR-IAs). Let Γ denote the DM's attention rule, where $\Gamma(S)$ represents the alternatives in S that attract her attention. As a consequence

[1] In this paper, without any further specification, we use "menus" for menus that contain mutually compatible alternatives.

of idempotence, applying Γ multiple times to S will yield the same result as the first application, i.e., $\Gamma(S) = \Gamma(\Gamma(S))$. From a behavioral standpoint, if the DM is unaware of certain alternatives, they should be treated as if they do not exist. Therefore, she should perceive S and $\Gamma(S)$ as identical.

It is worth noting that the idempotent attention rule is a generalization of attention filters introduced in Masatlioglu et al. (2012). Attention filters assume that the DM regards every menu T with $\Gamma(S) \subseteq T \subseteq S$ as identical. However, the idempotent attention rule allows for more complex attention patterns, enabling a better understanding of the interactions between alternatives in forming attention.

Our main contribution is to introduce idempotent attention rules. These rules allow the focus to be placed on the preferences of a subcollection, rather than on the preferences of the entire collection of menus. A major tool used in this paper is basic sets, where a set B is basic if no proper subset of B is indifferent to it. Under the AR-IA, the DM's attention on S can be regarded as an indifferent basic subset of S. By analyzing preferences among basic sets, we are able to retrieve both the DM's attention rule and utility function at the same time.

To illustrate the differences between idempotent attention rules and attention filters, we present the characterization of AR-IAs as well as additive representations under attention filter (AR-AFs). For the sake of simplicity, we assume that the utility function is non-negative. There is no difference in the way utility formations are conducted in the AR-IA and AR-AF. However, because attention filters impose more restrictions on attention formation, an additional axiom is necessary for characterizing the AR-AF. As expected, this additional axiom is also based on the preferences among basic sets.

The AR-IA also has practical implications. Two special cases of this model are introduced: the top k AR and the simple cardinality-based ordering under idempotent attention (SCO-IA). The top k AR assumes that a decision maker selects a fixed number of alternatives from a menu. It places a restriction on the cardinality of attention rules. In this spirit, idempotent attention rules can be viewed as the underlying selection procedure used by the DM when $\Gamma(S)$ determines the utility of S. The SCO-IA, on the other hand, assumes that the utility of each item is the same. It extends the ranking proposed by Pattanaik and Xu (1990), which reflects the degree of freedom of choice to the DM with limited attention.

The structure of this paper is as follows. Section 2 presents the model in a formal manner. Section 3 provides characterizations of AR-IAs and AR-AFs. In section 4, we discuss topics related to attention formation. Section 5 covers two applications of the AR-IA. Finally, Section 6 concludes the paper.

2 The Model

Let X be a nonempty finite set, and $\mathcal{X}$ be the collection of all nonempty subsets of X. We interpret X as a collection of all conceivable alternatives of interest and $\mathcal{X}$ as the collection of all menus that consist of mutually compatible alternatives. To simplify notation, we designate sets in a multiplication fashion hereafter. For example, a menu $\{x, y\}$ is denoted as xy in this paper.

The DM has a preference relation $\succcurlyeq$ on $\mathcal{X}$, which an outside researcher can observe. As usual, $\succcurlyeq$ is a binary relation on $\mathcal{X}$, that is, $\succcurlyeq \subseteq \mathcal{X} \times \mathcal{X}$. We write $S \succcurlyeq T$ instead of $(S, T) \in \succcurlyeq$, when the DM thinks S is weakly better than T. The strict preference $\succ$ and indifference $\sim$ are defined as: Given $S, T \in \mathcal{X}$, we say that $S \succ T$ if $S \succcurlyeq T$ and not $T \succcurlyeq S$, and $S \sim T$ if $S \succcurlyeq T$ and $T \succcurlyeq S$. $\succ$ and $\sim$, based on the definitions, they are the asymmetric and symmetric parts of $\succcurlyeq$, respectively.

Fishburn (1992) explores additive representations. Specifically, a preference relation $\succcurlyeq$ on $\mathcal{X}$ admits an **Additive Representation (AR)** if there exists $u : X \to \mathbb{R}$ such that for any $S, T \in \mathcal{X}$, $S \succcurlyeq T$ if and only if $\sum_{s \in S} u(s) \geq \sum_{t \in T} u(t)$. In this paper, $\sum_{s \in S} u(s)$ is referred to as the additive utility of S.

Unlike the indirect utility representation, the AR takes into account all the alternatives in a menu to determine its value. When the realization of each menu consists of multiple items, the AR provides a useful explanation for preference formation.

2.1 Additive Representations under Limited Attention

The AR suggests that DMs analyze all options in a menu to determine its value, which implies that they pay attention to all options.

Nevertheless, Masatlioglu et al. (2012) suggest that DMs pay attention to some of the alternatives in a menu because they have limited cognitive abilities. Several papers in Psychology have confirmed this statement and have demonstrated that there are some limits to the amount of information that can be stored even for a short period of time (e.g., Cowan (2001), Cowan et al. (2005)). Due to the limitations in cognitive capacity, DMs may intentionally or unintentionally overlook some alternatives in a menu (Broadbent (1958)).

Limited attention has significant implications for preference formation and decision-making. The failure to account for it may result in biased estimates and incorrect conclusions. As an example, the estimation of consumers' product-specific demand curves for PCs can be biased if full attention is assumed, while more realistic estimates can be obtained if limited attention is assumed (see Goeree (2008)).

In this paper, we will examine additive representations under limited attention.

2.1.1 Attention Rules

Compared to the AR, attention rules are the unique features of additive representations under limited attention. Among all possible attention rules, we focus on the idempotent ones.

Definition. *A mapping* $\Gamma : \mathcal{X} \to \mathcal{X}$ *is an* ***idempotent attention rule*** *if* $\Gamma(S) \subseteq S$ *and* $\Gamma(S) = \Gamma(\Gamma(S))$ *for all* $S \in \mathcal{X}$*.*

Given an arbitrary menu S, $\Gamma(S)$ is a nonempty subset of S, and it is the set of alternatives that catches the DM's attention.

Idempotent attention rules state that the DM pays attention to the same alternatives in S and $\Gamma(S)$. This means that the DM is not aware of the ignored options in a menu and cannot distinguish between S and $\Gamma(S)$. As a result, S and $\Gamma(S)$ are considered the same.

The idempotent attention rule generalizes a well-known attention rule used in Masatlioglu et al. (2012): attention filters.

Definition. *An attention rule* Γ *is an* ***attention filter*** *if for any* $S \in \mathcal{X}$, $\Gamma(T) = \Gamma(S)$ *whenever* $\Gamma(S) \subseteq T \subseteq S$.

Compared to idempotent attention rules, attention filters assume that the DM pays attention to the same alternatives in any subset T of S that contains $\Gamma(S)$. Thus, the DM considers all sets between $\Gamma(S)$ and S to be identical.

The differences between idempotent attention rules and attention filters stem from their assumptions about overlooked alternatives. Attention filters assume that overlooked alternatives are partially context-independent. It is assumed that the DM is unaware of the existence of some options in a menu S if her attention rule is an attention filter. Consequently, expanding $\Gamma(S)$ with any subset of these overlooked alternatives should not affect the DM's attention on S. Conversely, the idempotent attention rule allows for this alteration. For instance, in Figure 1, if S and T are two menus, and the DM pays attention to the alternatives in $\Gamma(S)$ and $\Gamma(T)$, respectively, adding some omitted alternatives from $T \setminus \Gamma(S)$ to $\Gamma(S)$ changes the attention. Therefore, Γ can be an idempotent attention rule but not an attention filter. This situation will be discussed in Section 4.3.

2.1.2 Additive Representations under Idempotent Attention

We investigate additive representations under idempotent attention (AR-IAs). Additionally, we study additive representations under attention filters (AR-AFs) to gain insight into the interactions between alternatives in forming attention.

To ease our comparison, we also assume that the utility function is nonnegatively valued.[2] This assumption simplifies the analysis by allowing the DM to freely discard any alternatives in a menu with negative utility. The DM can therefore concentrate on the "good" items rather than the "bad" ones on the menu.

[2]For the AR-IA, we can relax the assumption of nonnegative utility. The characterization has been provided in the Online Appendix.

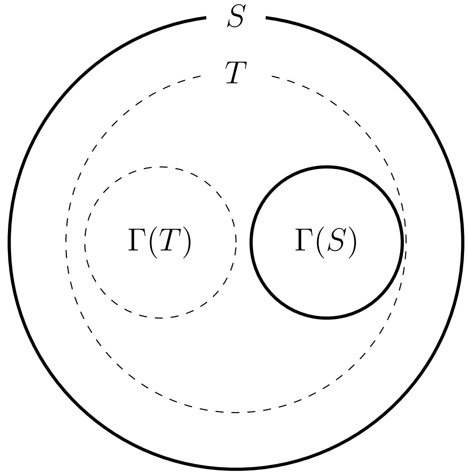

Figure 1: Idempotent Attention but not Attention Filter

Definition. *$\succsim$ on $\mathcal{X}$ admits an* ***Additive Representation under Idempotent Attention (AR-IA)*** *if there exist an idempotent attention rule $\Gamma : \mathcal{X} \to \mathcal{X}$, and a function $u : X \to \mathbb{R}^+$ such that for any $S, T \in \mathcal{X}$,*

$$S \succsim T \iff \sum_{s \in \Gamma(S)} u(s) \geq \sum_{t \in \Gamma(T)} u(t).$$

Given any pair of above Γ and u, we say that $\succsim$ on $\mathcal{X}$ admits an AR-IA under (Γ, u). Moreover, $\succsim$ on $\mathcal{X}$ admits an ***Additive Representation under Attention Filter (AR-AF)*** *under (Γ, u) if $\succsim$ on $\mathcal{X}$ admits an AR-IA under (Γ, u) where Γ is an attention filter.*

There is no doubt that if $\succsim$ on $\mathcal{X}$ admits ARs, then AR-AFs and AR-IAs must also be admitted. The converse, however, is not necessarily true.

Example 1. *Let $X = xyz$. $\succsim$ on $\mathcal{X}$ is $xy \succ xz \sim xyz \succ x \succ yz \succ y \succ z$. We can let $u(x) = 4, u(y) = 2, u(z) = 1$, and*

$$\Gamma(S) = \begin{cases} xz & \text{if } S = xyz, \\ S & \text{if } S \neq xyz. \end{cases}$$

It is easy to check that $\succcurlyeq$ *on* $\mathcal{X}$ *admits the AR-IA under* (Γ, u). *Since* Γ *is an attention filter,* $\succcurlyeq$ *on* $\mathcal{X}$ *also admits the AR-AF under* (Γ, u). *However,* $\succcurlyeq$ *on* $\mathcal{X}$ *does not admit an AR. To see this, we can suppose that it admits AR, and let* u *be the consistent utility function. Then* $yz \succ y$ *suggests that* $u(z) > 0$. *However,* $xy \sim xyz$ *implies that* $u(z) = 0$.

AR-AFs are special cases of AR-IAs. However, due to the difference between the idempotent attention rules and attention filters, some preferences on $\mathcal{X}$ admit AR-IAs but do not admit AR-AFs.

Example 2. *Let* $X = xyz$. $\succcurlyeq$ *on* $\mathcal{X}$ *is* $xy \succ xz \succ xyz \sim x \sim yz \succ y \succ z$. *Let* $u(x) = 3, u(y) = 2, u(z) = 1$, *and*

$$\Gamma(S) = \begin{cases} x & \text{if } S = xyz, \\ S & \text{if } S \neq xyz. \end{cases}$$

$\succcurlyeq$ *admits an AR-IA under* (Γ, u). *However,* $\succcurlyeq$ *does not admit an AR-AF. It is easy to check that if* $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-IAs,* $\Gamma(S)$ *should be one of the indifferent subsets of* S. *Hence, all subsets of* X *should catch full attention except* xyz. *For AR-AFs, we only need to consider* $\Gamma(xyz)$. *If* $\Gamma(xyz) = x$, *then* $\Gamma(xy) = \Gamma(xz) = x$ *by the definition of attention filters which cannot be true. If* $\Gamma(xyz) = xyz$, *then* $u(x) + u(y) + u(z) = u(x)$ *because* $xyz \sim x$. *Thus,* $u(z) < 0$.[3] *As a result,* $\succcurlyeq$ *does not admit an AR-AF.*

3 Characterizations

3.1 The Characterization of AR-IA

When $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA, $\succcurlyeq$ must be complete and transitive.

Axiom 1. *(WO: Weak Order)* $\succcurlyeq$ *on* $\mathcal{X}$ *is complete and transitive.*

[3]Despite relaxing the assumption of utility functions, $\succcurlyeq$ still does not admit an AR-AF. Since $xz \succ x$ implies that $u(x) + u(z) > u(x)$, i.e., $u(z) > 0$.

Based on the AR-IA, the utility of a menu S is equal to the additive utility of $\Gamma(S)$. The preference $\succcurlyeq$ induced by the AR-IA can be viewed as ranking menus in $\Gamma(\mathcal{X}) := \{\Gamma(S) : S \in \mathcal{X}\}$ by the AR.

The ranking on $\Gamma(\mathcal{X})$ can further be reduced to one of its subsets. Assume that $\succcurlyeq$ on $\mathcal{X}$ admits the AR-IA. Take any $S \in \mathcal{X}$, we know that $S \sim \Gamma(S)$. If there is a $\Gamma(T)$ where $\Gamma(T) \subset \Gamma(S)$ and $\Gamma(S) \sim \Gamma(T)$, the utility of S will equal the additive utility of $\Gamma(T)$. We can consider that the DM pays attention to $\Gamma(T)$ in S. However, this reduction process invalidates any $\Gamma(S)$ that processes no proper indifferent subset. Despite the fact that Γ is unobservable, we can derive this characteristic from $\succcurlyeq$ on $\mathcal{X}$.

Definition. *$S \in \mathcal{X}$ is **basic** if there is no $T \subset S$ such that $T \sim S$.*

The collection of basic sets is denoted by $\mathcal{B}$. A basic set B is a corresponding basic set of S if $B \subseteq S$ and $S \sim B$. $\mathcal{B}(S)$ refers to the collection of corresponding basic sets of S.

The concept of basic sets arises from the intuition of reducing a set to its smallest indifferent subset based on Γ. $\Gamma(\Gamma(S)) = \Gamma(S)$ indicates that the DM pays attention to all alternatives in $\Gamma(S)$. As a result, if we wish to interpret a basic set B as the $\Gamma(S)$ for some S with $B \subseteq S$, the DM must pay full attention to B.

Proposition 1. *If $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA under (Γ, u), then $\Gamma(B) = B$ for all $B \in \mathcal{B}$.*

Proof. Let $\succcurlyeq$ be a preference on $\mathcal{X}$ that admits an AR-IA, and (Γ, u) be a pair of the corresponding idempotent attention rule and utility function. By contradiction, suppose that there is a basic set B such that $B \neq \Gamma(B)$. That is, $\Gamma(B) \subset B$. As Γ is idempotent, we have $\Gamma(B) = \Gamma(\Gamma(B))$. The additive utility of $\Gamma(\Gamma(B))$ equals the additive utility of $\Gamma(B)$. Hence, $B \sim \Gamma(B)$, which implies that $B \notin \mathcal{B}$. □

Rather than directly investigating $\succcurlyeq$ on $\mathcal{X}$, we can focus on $\succcurlyeq$ on $\mathcal{B}$. Because basic sets must attract full attention, $\succcurlyeq$ on the $\mathcal{B}$ admits an AR. Consider any $B_1, B_2 \in \mathcal{B}$ with $B_1 \succcurlyeq B_2$. We know that $\sum_{b \in B_1} u(b) \geq \sum_{b \in B_2} u(b)$. Since $\mathcal{B} \subseteq \mathcal{X}$, the utility consistent with the AR-IA must be a solution to the system of inequalities induced by $\succcurlyeq$ on basic sets. The condition for the existence of solutions to a finite system of linear inequalities can be found in Kraft et al. (1959), Scott (1964), Krantz et al. (1971) and Fishburn (1992).

Axiom 2 (NR: Nonnegative Remainders)**.** *There do not exist a positive integer m and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \succcurlyeq T_n$ for all n, and $S_n \succ T_n$ for some n, such that $\sum_{n=1}^m \mathbb{1}_{S_n}(x) \leq \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \in X$ where $\mathbb{1}$ is the indicator function.*

NR can be interpreted as a stricter condition of the Monotonicity axiom on basic sets: $B_1 \supseteq B_2 \implies B_1 \succcurlyeq B_2$ for any $B_1, B_2 \in \mathcal{B}$.[4] In particular, when $m = 1$, NR combined with the completeness of $\succcurlyeq$ is equivalent to the Monotonicity axiom on basic sets. To see this, let $m = 1$. In this case, NR asserts that there is no $B_1, B_2 \in \mathcal{B}$ with $B_2 \succ B_1$ such that $\mathbb{1}_{B_2}(x) \leq \mathbb{1}_{B_1}(x)$ for all $x \in X$. It is equivalent to saying that $B_2 \subseteq B_1$ implies that B_2 is not strictly preferred to B_1. By combining the completeness of $\succcurlyeq$ on $\mathcal{B}$, we are able to derive the Monotonicity axiom.

However, WO and Monotonocity on $\mathcal{B}$ can not guarantee that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA.

Example 3. *Let $X = xyz$ and $\succcurlyeq$ on $\mathcal{S}$ is $xyz \succ xz \succ xy \succ x \succ yz \succ y \succ z$. We first observe that all the sets are basic, and $\succcurlyeq$ on $\mathcal{X}$ satisfies WO and Monotonicity. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA under (Γ, u), we then have*

$$xz \succ xy \iff u(x) + u(z) > u(x) + u(y) \iff z \succ y.$$

However, we have $y \succ z$. To see whether it violates NR, we can consider $\{xz, y\}$ and $\{xy, z\}$. We know that $xz \succ xy$ and $y \succ z$, but x, y, z shows one time in both collections.

NR can be interpreted as a generalization of Monotonicity. Take any two collections of basic sets $\{S_n\}_{n=1}^m$ and $\{T_n\}_{n=1}^m$, we refer that $\{S_n\}_{n=1}^m$ contains less content than $\{T_n\}_{n=1}^m$ if the former collection contains less x than the latter one for every

[4] Later, we will investigate an application called Borda-IA. There will be a more clear relationship between NR and the Monotonicity axiom.

$x \in X$. Equivalently, $\sum_{n=1}^{m} \mathbb{1}_{S_n}(x) \leq \sum_{n=1}^{m} \mathbb{1}_{T_n}(x)$ for all $x \in X$. According to the Monotonicity Axiom, the DM cannot prefer a smaller set. Analogously, NR requires that the DM cannot prefer $\{S_n\}_{n=1}^{m}$ to $\{T_n\}_{n=1}^{m}$, i.e., the preference cannot be $S_n \succcurlyeq T_n$ for all n and $S_n \succ T_n$ for some n.

WO and NR are the axioms that we need to characterize the AR-IA.

Theorem 1. *$\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies WO and NR.*

Proof. See Appendix A. □

Due to the fact that singleton sets are basic, WO and NR guarantee the existence of a utility function that is consistent with the AR-IA on basic sets. We can associate a set S with one of its corresponding basic sets. First, we give a permutation on $\mathcal{B}$ and let $\mathcal{B} = \{B_n\}_{n=1}^{m}$. Then, let $\Gamma(S) = B_{n^*}$ where $n^* = \min\{n : B_n \subseteq S \textit{ and } B_n \sim S\}$. In light of the fact that the basic sets capture full attention, it is apparent that Γ is idempotent. Consequently, in the context of the AR-IA, the DM believes that a collection of menus containing more content is better, despite not paying attention to the items within each menu.

3.2 The Characterization of AR-AF

The difference between AR-IAs and AR-AFs arises from the distinction between idempotent attention rules and attention filters. Due to the fact that AR-AFs are special cases of AR-IAs, and WO and NR guarantee the existence of utility functions, we need only consider the requirements of attention filters. [5]

The AR-IA permits us to assign $\Gamma(S) = B$ for one of the corresponding basic sets B of S. In contrast, under the AR-AF, $\Gamma(S)$ must be a non-basic set for some $S \in \mathcal{X}$, given the preference relation $\succcurlyeq$.

[5] It should be noted that this logic is also used to characterize the AR-IA. Utility functions are guaranteed to exist by WO and NR. As a requirement of the idempotent attention rule, there must be an indifferent subset of S that attracts the full attention of the DM. The definition of basic sets includes this requirement. That is, $\mathcal{B}(S) \neq \emptyset$ for all $S \in \mathcal{X}$.

Example 4. *Let* $X = xyz$*.*$\succcurlyeq$ *on* $\mathcal{X}$ *is* $xy \succ xyz \sim xz \sim x \succ yz \sim y \succ z$*. It is easy to check that* $\succcurlyeq$ *on* $\mathcal{X}$ *admits an AR-IA. For* $u(x) = 2, u(y) = 1, u(z) = 0$*, let's consider the following* Γ_1 *and* Γ_2*:*

$$\Gamma_1(S) = \begin{cases} xz & \text{if } S = xyz, \\ S & \text{if } S \neq xyz. \end{cases}, \quad \text{and} \quad \Gamma_2(S) = \begin{cases} x & \text{if } S = xyz, \\ S & \text{if } S \neq xyz. \end{cases}.$$

Γ_1 *is an attention filer, while* Γ_2 *is not an attention filter. Moreover, we cannot have* $\Gamma(xyz) = x$ *for any AR-AFs, implying that* $\Gamma(xyz)$ *cannot be any* xyz*'s corresponding basic set.*

In the context of the AR-AF, the set of alternatives with zero utility plays a crucial role. If $\Gamma(S)$ is an indifferent non-basic proper subset of S, then the set difference between $\Gamma(S)$ and any of S's corresponding basic sets must have zero utility. Under the assumption of nonnegative utility, these alternatives must be ranked as the least preferred alternatives. We use $MIN(X, \succcurlyeq) := \{x \in X : y \succcurlyeq x \text{ for all } y \in X\}$ to denote them. It should be noted that the dominated alternative may not necessarily have zero utility. For instance, consider the preference relation $xy \succ x \succ y$. This preference relation admits AR-AFs, and it is true that $u(y) > 0$ for all consistent utility functions. We provide a condition to characterize the existence of null alternatives.

Condition 1 (ENS: Existence of Null Sets)**.** *There do not exist a positive integer* m *and* $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ *where* $S_n \succcurlyeq T_n$ *for all* n*, and* $S_n \succ T_n$ *for some* n*, such that* $\sum_{x \in MIN(X,\succcurlyeq)} \sum_{n=1}^m \mathbb{1}_{S_n}(x) > \sum_{x \in MIN(X,\succcurlyeq)} \sum_{n=1}^m \mathbb{1}_{T_n}(x)$*, and* $\sum_{n=1}^m \mathbb{1}_{S_n}(x) = \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ *for all* $x \notin MIN(X, \succcurlyeq)$*.*

When $m = 1$, ENS becomes that there are no basic sets S and T with $S \succ T$ such that $T \subset S$ and $S \setminus T \subseteq MIN(X, \succcurlyeq)$. In other words, if the DM strictly prefers a larger menu, then the reason cannot be that the larger menu contains more dominated alternatives. Otherwise, the dominated alternatives should be of positive utility to the DM. However, the $m = 1$ case is not sufficient to characterize the existence of null alternatives under the AR-IA.

Example 5. *Let $X = abcd$. The preference on $\mathcal{B}$ is $bcd \succ ab \succ a \succ b \succ c \succ d$.*[6] *It is easy to check that this preference satisfies NR. Moreover, there are no basic sets S and T such that $T \subset S$ and $S \setminus T \subseteq MIN(X, \succcurlyeq)$. $bcd \succ ab$ implies that $u(c) + u(d) > u(a)$ for all consistent u. We then have $u(d) > 0$ for all consistent u because $a \succ c$.*

As with NR, ENS posits a consistent condition for evaluating basic sets. Assume that the DM assigns zero utility to the dominated alternative. According to the AR-IA, the DM cannot strictly prefer one collection of basic sets over another if the preferred collection only contains more dominated alternatives than the other collection.

ENS refines the solution sets to the linear inequalities induced by $\succcurlyeq$ on $\mathcal{B}$. It identifies when the dominated alternatives can have zero utility.

Proposition 2. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. There exists a consistent u such that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$ if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies ENS.*

Proof. See Appendix B. □

We then denote the collection of alternatives that may have zero utility as N, and infer N as the set of null alternatives.

$$N = \begin{cases} MIN(X, \succcurlyeq) & \text{if } \succcurlyeq \text{ satisfies ENS,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Axiom 3 (IB: Indifferent Betweenness)**.** *For each $S \in \mathcal{X}$ there exist a set $B \in \mathcal{B}(S)$ and a $T \in \mathcal{X}$ where $B \subseteq T \subseteq S$ and $T \setminus B \subseteq N$, such that $T \sim Y \sim S \sim B$ for all Y with $T \subseteq Y \subseteq S$.*

[6] We slightly abuse the concept of basic sets. Because basic sets are defined as sets that satisfy some conditions on preference, the preference given in this example should be understood as a part of the preference on $\mathcal{X}$. In this example, the preference can be extended to the preference on $\mathcal{X}$ in which the sets listed are basic. To see this, for a "non-basic" set S, we can let $S \sim a$ if $a \in S$, $S \sim b$ if $a \notin S$ and $b \in S$, $S \sim c$ if $a, b \notin S$ and $c \in S$, and $S \sim d$ if $a, b, c \notin S$ and $d \in S$.

Let $\mathcal{IB}(S)$ be the collection of $T \in \mathcal{X}$ such that: (i) for all Y with $T \subseteq Y \subseteq S$, $T \sim Y \sim S$, and (ii) there is a $B \in \mathcal{B}(S)$ with $B \subseteq T$ such that $T \setminus B \subseteq N$.

IB relates to the requirements of attention filters. For AR-AFs, we only need to select the pairs of (Γ, u) where Γ is an attention filter from all consistent pairs of AR-IAs. We know that the DM considers all alternatives in a basic set under AR-IAs, and this also applies to AR-AFs. For any non-basic set S, $\mathcal{B}(S)$ is nonempty. When one of $B \in \mathcal{B}(S)$ satisfies that $T \sim B \sim S$ for all $B \subseteq T \subseteq S$, then we can let $\Gamma(S) = B$. Consider the case in which there is no such basic set of S. IB implies that there is an indifferent subset T of S such that T only contains extra null alternatives compared to one of S's corresponding basic sets. We then can let $\Gamma(S) = T$. This procedure can be simplified when all alternatives have strictly positive utility. In this case, $N = \emptyset$. IB then becomes that for every non-basic set S, there is a basic set $B \in \mathcal{B}(S)$ such that all sets in between S and B are indifferent.

It turns out that WO, NR and IB can characterize the AR-AF.[7] Formally,

Theorem 2. *$\succcurlyeq$ on $\mathcal{X}$ admits an AR-AF if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies WO, NR and IB.*

Proof. See Appendix C. □

3.3 Uniqueness of Representations

In the context of both AR-IAs and AR-AFs, it is important to note that there exist multiple pairs of (Γ, u) that represents the same $\succcurlyeq$. For instance, if u is a consistent utility function of the AR-IA or AR-AF under Γ, then (cu, Γ) with $c > 0$ will also be a consistent pair of the AR-IA or AR-AF.[8]

In this section, we focus on the uniqueness of utility functions and attention rules for $\succcurlyeq$ on $\mathcal{X}$ that admits either AR-IAs or AR-AFs.[9] Henceforth, we use the notation $\{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ and $\{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ to represent the collection of consistent pairs for the AR-IA and the AR-AF, respectively.

[7]The checking of independence of these three axioms is attached in Appendix D.

[8]If the converse holds, the utility function is said to be unique under similarity transformation (see Fishburn (1992)). In the Online Appendix, we provide several results related to it.

[9]In the Online Appendix, we also investigate the uniqueness of the consistent pairs (Γ, u).

3.3.1 Uniqueness of Utility Functions

The utility function is a solution to the system of linear inequalities induced by the preference over $\mathcal{B}$. Since the singletons are basic, the collection of possible solutions $\mathcal{U}_{\succcurlyeq}$ is defined as follows:

$$\mathcal{U}_{\succcurlyeq} = \left\{ u \in \mathbb{R}^X_+ : \sum_{b \in B_1} u(b) \geq \sum_{b \in B_2} u(b) \text{ for all } B_1, B_2 \in \mathcal{B} \text{ with } B_1 \succcurlyeq B_2 \right\}. \quad (1)$$

Corollary 3. *Assume $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. For any $u : X \to \mathbb{R}^+$, there is an idempotent attention rule Γ_i such that $(\Gamma_i, u) \in \{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ if and only if $u \in \mathcal{U}_{\succcurlyeq}$.*

Proof. Assume $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. As basic sets must catch full attention, $\succcurlyeq$ on $\mathcal{B}$ admits the AR when $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Then the consistent u must in $\mathcal{U}_{\succcurlyeq}$. For the converse direction, take any $u \in \mathcal{U}_{\succcurlyeq}$. Similar to the construction in proving the Theorem 1, we can associate every S to one of its corresponding basic sets B. Let $\Gamma_i(S) = B$. It is evident that $(\Gamma_i, u) \in \{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$. □

For the AR-AF, even though the evaluation of a menu S equals to every $B \in \mathcal{B}(S)$, for some $u \in \mathcal{U}_{\succcurlyeq}$ we may not be able to find an attention filter Γ_j such that $(\Gamma_j, u) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$. As in the Example 4, the collection of basic sets is $\{xy, x, y, z\}$. It is easy to verify that $u(x) = 3$, $u(y) = 2$ and $u(z) = 1$ is a solution to the system of linear inequalities induced by the preference on $\mathcal{B}$. However, there is no attention filter Γ_j such that (Γ_j, u) is a consistent pair of the AR-AF because we are unable to locate a suitable image for $\Gamma_j(xyz)$.

Corollary 4. *Assume $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. For a $u : X \to \mathbb{R}^+$, if there is an attention filter Γ_j such that $(\Gamma_j, u) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ then $u \in \mathcal{U}_{\succcurlyeq}$. The converse holds when $N = \emptyset$.*

Proof. Assume $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. $\succcurlyeq$ on $\mathcal{X}$ also admits AR-IAs, thus we have the first half of the statement by Corollary 3. For the second half of the statement, assume that $N = \emptyset$. We then know that for any $S \in \mathcal{X}$, there is a $B \in \mathcal{B}(S)$ such that for all Y with $B \subseteq Y \subseteq S$, $Y \sim B \sim S$. By the similar construction as in the proof of Theorem 2, we can have an attention filter Γ_j where $\Gamma_j(\mathcal{X}) = \mathcal{B}$. Further, $(\Gamma_j, u) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ for all $u \in \mathcal{U}_{\succcurlyeq}$. □

3.3.2 Uniqueness of Attention Rules

When all consistent pairs share the same attention rule, we can pinpoint the DM's attention for sure. We infer this case as the attention rule is unique.

Definition. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. The idempotent attention rule Γ_i is **unique** if $\Gamma_i = \Gamma_{i'}$ for all $i, i' \in I_{\succcurlyeq}$. Similarly, suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. The attention filter Γ_j is **unique** if $\Gamma_j = \Gamma_{j'}$ for all $j, j' \in J_{\succcurlyeq}$.*

In the AR-IA, we can treat a corresponding basic set B of S as $\Gamma(S)$. Furthermore, if S contains an alternative x where $u(x) = 0$ and $x \notin B$, then a consistent attention filter Γ' exists where $\Gamma'(S) = B \cup \{x\}$. Thus, the uniqueness of idempotent attention rules is determined by both the number of corresponding basic sets and the existence of null alternatives.

Proposition 3. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA. The idempotent attention is unique if and only if*

(i) for every $S \in \mathcal{X}$ there is a unique $B \in \mathcal{B}$ such that $B \subseteq S$ and $B \sim S$, i.e., $|\mathcal{B}(S)| = 1$ for all $S \in \mathcal{X}$.

(ii) $\succcurlyeq$ on $\mathcal{X}$ does not satisfy ENS, i.e., $N = \emptyset$.

Proof. Suppose that $\succcurlyeq$ on $\mathcal{X}$ satisfy (i) and (ii). Take any $S \in \mathcal{X}$; if S is basic, we know that $\Gamma_i(S) = S$ for all $i \in I$. If S is not basic, then we know there is a $\Gamma_i(S) = B$ where $\{B\} = \mathcal{B}(S)$. By contradiction, suppose that there is another Γ_i where $\Gamma'_i(S) = T$ where $T \neq B$. By (i), we have $T \notin \mathcal{B}$ which implies that $N \neq \emptyset$. For the converse direction, suppose that the idempotent attention rule is unique. (i) is obvious from the proof of Theorem 1. Suppose that (ii) does not hold, then we know that there is an $i \in I$ such that $u_i(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. Take any $y \neq x$, we can let $\Gamma'_i(xy) = xy$ and $\Gamma'_i(S) = \Gamma_i(S)$ for all $S \neq xy$. It is clear that (Γ'_i, u_i) is still a consistent AR-IA pair. □

The attention filter is a special case of the idempotent attention rule. From the proof of Theorem 2, we can interpret the T in $\mathcal{IB}(S)$ as a candidate of $\Gamma_j(S)$ where $j \in J_{\succcurlyeq}$.

Proposition 4. *Suppose that* $\succcurlyeq$ *on* $\mathcal{X}$ *admits an AR-IA. The attention filter is unique if and only if*

(i) for any S *there is a unique basic set* B *such that* $T \sim S \sim B$ *for all* T *with* $B \subseteq T \subseteq S$.

(ii) $\succcurlyeq$ *on* $\mathcal{X}$ *does not satisfy ENS, i.e.,* $N = \emptyset$.

Proof. By the proof of Theorem 2, we know that the attention filter Γ_j is unique if and only if $|\mathcal{IB}(S)| = 1$ for all $S \in \mathcal{X}$. The sufficiency of (i) and (ii) are relatively straightforward. For the necessity, we can assume that $|\mathcal{IB}(S)| = 1$ for all $S \in \mathcal{X}$. We only need to prove the necessity of (ii). By contradiction, suppose that $N = MIN(X, \succcurlyeq)$. Take any $x, y \in X$ where $x \in MIN(X, \succcurlyeq)$. We have either $x \sim xy$ or $y \sim xy$ or both which suggests that $|\mathcal{IB}(xy)| > 1$. □

3.4 Comments

In the AR-IA, we can also drop the assumption of the nonnegative utility function. We still require the induced system of linear inequalities to have a solution when we characterize AR-IAs without the assumption of u. The NR axiom can be replaced by the finite cancellation axiom (see Kraft et al. (1959), Scott (1964), Krantz et al. (1971), and Fishburn (1992)) in characterizing the AR-IA (see Online Appendix). However, in the AR-AF, we need to characterize the potential null sets with respect to additive utility. The assumption of the nonnegative utility function simplifies our characterization of the AR-IA by describing it in IB.

Sometimes, we are unable to observe the DM's preference over all menus. Instead, we can only have the preference on a subset $\mathcal{M}$ of $\mathcal{X}$. To characterize the AR-IA in this case, we need to know that for any $S \in \mathcal{M}$, there is a $T \subseteq S$ such that there is no $Y \subset T$ where $T \sim Y \sim S$. In other words, we should at least be able to identify one of the corresponding basic sets of S. Therefore, in order to apply these theorems in practice, we must acquire sufficient data.

4 Revealed Attention

Identifying the DM's attention in each menu is a crucial step as there exist multiple attention rules that are consistent with our representations. This enables us to compare idempotent attention rules with attention filters and gain insight into the interactions among alternatives that shape the DM's attention.

4.1 Revealed Attention under AR-IAs

Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Even though there are infinite consistent pairs of (Γ, u) that represent the same preference, there are only a finite number of idempotent attention rules because $\mathcal{X}$ is finite. The DM's attention rule can be any one of them.

For AR-IAs, we can consider a $B \in \mathcal{B}(S)$ as the DM's attention on S. It is important to note that a consistent idempotent attention rule Γ_i may not map S to one of its corresponding basic sets (see Example 4). It is also possible for a non-basic indifferent subset T of S to represent the attention on S if T only contains extra null alternatives compared to a corresponding basic set of S.

Proposition 5. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. For any $S \in \mathcal{X}$, $\Gamma_i(S) = T$ for some $i \in I_{\succcurlyeq}$ if and only if $T \sim B$ and $T \setminus B \subseteq S \cap N$ for some $B \in \mathcal{B}(S)$.*

Proof. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Take any S and Γ_i with $i \in I_{\succcurlyeq}$. Let $\Gamma_i(S) = T$. If $T \in \mathcal{B}$, we are done. If $T \notin \mathcal{B}$, we know that there is a $B \in \mathcal{B}(S)$ such that $T \sim B$ and $T \setminus B \subseteq S \cap N$. For the converse direction, take any $T \in \mathcal{X}$ such that $T \sim B$ and $T \setminus B \subseteq S \cap N$ for a $B \in \mathcal{B}(S)$. Without loss of generality, suppose that $T \in \mathcal{B}^c$. By the proof of Theorem 1, there is $(\Gamma_i, u_i) \in \{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ where $\Gamma(\mathcal{X}) = \mathcal{B}$ such that $\Gamma_i(S) = B$, and $u_i(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. Let's consider

$$\Gamma_i'(Y) = \begin{cases} T & \text{if } Y = T \text{ or } S, \\ \Gamma_i(Y) & \text{otherwise.} \end{cases}$$

Let's first verify that Γ_i' is an idempotent attention rule. We first notice that $S \in \mathcal{B}^c$ as $T \in \mathcal{B}^c$. Therefore, there is no $Y \in \mathcal{X}$ such that $\Gamma_i(Y) = S$. Take any $Y \in \mathcal{X}$, if $Y = T$ *or* S, then $\Gamma_i'(Y) = \Gamma_i'(\Gamma_i'(Y))$. If $Y \neq T$ or $Y \neq S$, then $\Gamma_i'(\Gamma_i'(Y)) = \Gamma_i'(\Gamma_i(Y)) = \Gamma_i(Y) = \Gamma_i'(Y)$. Hence, Γ_i' is an idempotent attention rule. It is easy to verify that $(\Gamma_i', u_i) \in \{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$. □

Due to the possibility of multiple consistent idempotent attention rules for $\succcurlyeq$ on $\mathcal{X}$, it is difficult to determine which rule is adopted by the DM. To obtain a correct welfare implication, we use the strongest criterion of revealing attention as described in Masatlioglu et al. (2012). To put it another way, we are trying to determine under what circumstances an alternative and a menu must catch the full attention of the DM.

Definition. *Suppose that the $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. An alternative s* ***catches attention under AR-IAs*** *in S if $s \in \Gamma_i(S)$ for all $i \in I_{\succcurlyeq}$. A menu S* ***catches full attention under AR-IAs*** *if s catches attention in S for all $s \in S$, i.e., $\Gamma_i(S) = S$ for all $i \in I_{\succcurlyeq}$.*

In the AR-IA, we can restrict $\Gamma(\mathcal{X}) = \mathcal{B}$. As a result, if an alternative s catches attention in S, it must be contained in all $B \in \mathcal{B}(S)$. The converse is also true.

Corollary 5. *An alternative s catches attention under AR-IAs in S if and only if $s \in \bigcap_{B \in \mathcal{B}(S)} B$.*

Proof. Let $\bar{\mathcal{B}}(S) := \{T \subseteq S : \exists B \in \mathcal{B}(S) \text{ s.t. } T \sim B \text{ and } T \setminus B \subseteq N\}$. By Proposition 5, an alternative s catches attention under AR-IAs in S if and only if $s \in \bigcap_{T \in \bar{\mathcal{B}}(S)} T$. If $T \in \bar{\mathcal{B}}(S)$ is not a basic set, then there is a $B \in \mathcal{B}(S)$ such that $T \setminus B \subseteq S \cap N$. As a result, $s \in \bigcap_{T \in \bar{\mathcal{B}}(S)} T$ if and only if $s \in \bigcap_{B \in \mathcal{B}(S)} B$. □

Under AR-IAs, the DM considers all alternatives in any basic set. For any non-basic set S, we can associate it with one of its corresponding basic sets to construct a consistent idempotent attention rule. Therefore, the collection of menus that catch the DM's full attention coincides with $\mathcal{B}$.

Corollary 6. *A menu S catches full attention under AR-IAs if and only if S is basic.*

Proof. We first know that basic sets catch full attention under AR-IAs by Proposition 1. Suppose that S strongly catches full attention, and S is nonbasic. Then, we know that there is a corresponding basic set B of S that $S \setminus B \subseteq N$. As a result, there is an idempotent attention Γ consistent with AR-IAs such that $\Gamma(S) = B$. □

4.2 Revealed Attention under AR-AFs

We now assume that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. As shown in the proof of Theorem 2, we can associate S to $T \in \mathcal{IB}(S)$, and interpret T as the DM's attention on S.

Proposition 6. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. For any $S \in \mathcal{X}$, $\Gamma_j(S) = T$ for some $j \in J_{\succcurlyeq}$ if and only if $T \in \mathcal{IB}(S)$.*

Proof. The if part of this statement can be seen from the proof of Theorem 2. For another direction, take any $(\Gamma_j, u_j) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ where $\Gamma_j(S) = T$. If $T \in \mathcal{B}$, we are done. If $T \notin \mathcal{B}$, then we know that there is a $B \in \mathcal{B}(S)$ such that $T \setminus B \subseteq N$. Combining with the fact that $T \sim Y \sim S$ for all Y with $T \subseteq Y \subseteq S$, we have $T \in \mathcal{IB}(S)$. □

We also use the strongest criterion for revealing attention under AR-AFs.

Definition. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. An alternative s* ***catches attention under AR-AFs*** *in S if $s \in \Gamma_j(S)$ for all $j \in J_{\succcurlyeq}$. A menu S* ***catches full attention under AR-AFs*** *if s catches attention in S for all $s \in S$, i.e., $\Gamma_j(S) = S$ for all $j \in J_{\succcurlyeq}$.*

IB is the only difference between the characterization of AR-IAs and AR-AFs. For any set S, it is possible to find an indifferent subset T such that all the sets inbetween S and T are indifferent. As we construct attention filters, we are aware that any $T \in \mathcal{IB}(S)$ can serve as $\Gamma_j(S)$ for some $j \in J$.

Corollary 7. *An alternative s catches attention in S under AR-AFs if and only if $s \in \bigcap_{T \in \mathcal{IB}(S)} T$.*

For any menu S, we know $\mathcal{IB}(S)$ contains all the candidates for the DM's attention on S when Γ is an attention filter. When we are unable to find a proper subset T of S such that all sets lying between them are indifferent, then $\mathcal{IB}(S)$ must equal S under AR-AFs. We refer to such sets as weakly basic sets.

Definition. *A menu S is **weakly basic** if there is no $T \subset S$ such that $T \sim Y \sim S$ for all Y with $T \subseteq Y \subseteq S$.*

In the case of any weakly basic set S, any attention filers Γ must satisfy the condition $\Gamma(S) = S$, since S has no proper subset T, such that all the sets lying in between S and T are indifferent. To put it another way, weakly basic sets must attract full attention under AR-AFs. The converse is also true.

Corollary 8. *A menu S catches full attention under AR-AFs if and only if S is weakly basic.*

Proof. Take any menu S where it catches full attention under AR-AFs. By contradiction, we can assume that S is not weakly basic. Since S is not weakly basic, there is a proper subset T of S that $T \in \mathcal{IB}(S)$ by IB axiom. Hence, we know there is a Γ'_j that is consistent with AR-AFs such that $\Gamma'_j(S) = T$. For the converse direction, suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-AF under (Γ_j, u_j). By contradiction, suppose that $\Gamma_j(S) = T$ where $T \subset S$, and S is weakly basic. Then, we know that $\Gamma_j(Y) = T$ for any Y with $T \subseteq Y \subseteq S$ implies that $T \sim Y \sim S$. S, as a result, is not weakly basic. $\square$

4.3 Implications of Attention Formation

As special cases of idempotent attention rules, attention filters impose stricter conditions on DMs' attention. Specifically, they assume that removing any omitted alternatives from a menu will not alter the DM's attention. In contrast, idempotent attention rules allow for this possibility. In Figure 1, the alternatives in $S \setminus T$ make the DM pay attention to some alternatives in $S \setminus \Gamma(S)$ and omit $\Gamma(S)$.

Definition. *Suppose that* $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-IAs. For any* $S, T, Y \in \mathcal{X}$ *with* $Y, T \subset S$, Y ***obstructs the DM's attention*** *on* T *in* S *if for all* $i \in I$, $T \cap \Gamma_i(S) = \emptyset$ *and* $T \subseteq \Gamma_i(S \setminus Y)$.

Similar to revealing attention, we can focus on the collection of basic sets. The following corollary is a direct result of Proposition 5.

Corollary 9. *Suppose that* $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-IAs. For any* $S, T, Y \in \mathcal{X}$ *with* $Y, T \subset S$, Y *obstructs the DM's attention on* T *in* S *if and only if* $T \cap \bar{B} = \emptyset$ *for all* $\bar{B} \in \bar{\mathcal{B}}(S)$, *and* $T \subseteq B'$ *for all* $B' \in \mathcal{B}(S \setminus Y)$.

The effects of obstructions provide valuable insights for menu design. Suppose Y obstructs the DM's attention on T in S. When a menu provider wants to emphasize the existence of T for a menu S, rather than offering a larger menu S, he can provide a smaller menu $S \setminus Y$. Additionally, if Y hinders the DM's attention on T for each S with $Y, T \subset S$, menu providers who want DMs to focus on T should avoid providing a menu that contains the obstructing subset Y.

The implications of the idempotent attention rule on menu design extend beyond the obstruction effects. In the context of a menu S provided by a menu provider, there are three ways in which the provider can improve the menu's effectiveness: (i) providing a basic set $B \in \mathcal{B}(S)$ directly, rather than S; (ii) if there exists a subset T of S such that $T \succ S$, removing $S \setminus T$ from S to improve the DM's evaluation; and (iii) making the best alternatives in S more noticeable by using advertisements or other visual aids, if the DM pays no attention to them.

4.4 Comments

The practices of revealed attention provide a reliable way to infer DMs' attention on each menu. As a result, it facilitates the smallest collection of DMs' attention at the expense of potential candidates. It should be noted, however, that both Propositions 5 and Proposition 6 could provide potential representations of attention rules. It is possible to construct the weakest criteria to reveal attention under AR-IAs and AR-AFs.

For the AR-IA, an alternative s may attract the attention of the DM when and only when it belongs to T for some $T \in \bar{\mathcal{B}}(S)$. The AR-AF counterpart can be obtained by replacing $\bar{\mathcal{B}}(S)$ with $\mathcal{IB}(S)$. Under both the AR-IA and AR-AF, a menu S may capture the DM's attention if and only if $S \setminus B \subseteq N$ for some $B \in \mathcal{B}(S)$. The formal discussion can be found in the Online Appendix.

The weakest and strongest criteria of revealed attention give a range for the DM's attention given S. Furthermore, it enables us to compare interpersonal attention. In the Online Appendix, we use the most stringent criteria to accomplish this. Specifically, for any $S \in \mathcal{X}$, if DM 2 potentially pays attention to an alternative s then DM 1 must pay attention to it, then DM 1 is definitely more attentive than DM 2. It differs from the concept of capacity used in Geng and Özbay (2021) and Geng (2022) in that it emphasizes the heterogeneous role that alternatives play in forming DMs' attention.

5 Applications of the AR-IA

The AR-IA describes the aggregation behavior of the limitedly attentive DM. It is sometimes necessary to place additional restrictions in practice. This section examines two of them: First, the DM selects at most k alternatives from a menu; Second, the utility function is fixed.

5.1 Top k Alternatives

Γ can also be viewed as a selection process. The idempotence property of Γ implies that, in terms of preferences, the menu S is indifferent to $\Gamma(S)$. When $\Gamma(S)$ is the best alternative in S, this interpretation is consistent with the indirect utility representation. It is also possible to extend this framework to allow for the selection of multiple alternatives from a menu, which has been studied in many papers. For example, Eliaz et al. (2011) developed a model for choosing two alternatives, while Chambers and Yenmez (2018) studied the behavior of selecting the top k alternatives.

Definition. *$\succcurlyeq$ on $\mathcal{X}$ admits a **Top k AR** if there is a utility function $u : X \to \mathbb{R}^+$*

such that for all $S, T \in \mathcal{X}$

$$S \succcurlyeq T \iff \max_{\substack{S' \subseteq S \\ |S'| \leq k}} \sum_{s \in S'} u(s) \geq \max_{\substack{T' \subseteq T \\ |T'| \leq k}} \sum_{t \in T'} u(t).$$

Given an arbitrary permutation of $X = \{x_1, \ldots, x_{|X|}\}$ such that $\{x_i\} \succcurlyeq \{x_j\}$ for all $i \leq j$. Every $S \in \mathcal{X}$ can be considered as a subsequence of X after adopting this permutation. We denote S as $\{x_{S_1}, \ldots, x_{S_{|S|}}\}$.

Axiom 4. ***(k-Cutoff)*** *For any S with $|S| \leq k$, there is a $B \in \mathcal{B}(S)$ such that $S \setminus B \subseteq N$. Moreover, for any S with $|S| > k$, S is indifferent to $\{x_{S_n}\}_{n=i}^{k}$.*

It is impossible to distinguish whether the decision maker selects the top k alternatives or only pays attention to them under the AR-IA framework. Consequently, the top k AR is a particular case of AR-IAs.

Proposition 7. *$\succcurlyeq$ on $\mathcal{X}$ admits a Top k AR if and only if $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA, and $\succcurlyeq$ on $\mathcal{X}$ satisfies k-Cutoff.*

Proof. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits a top k AR under u. Consider an attention rule

$$\Gamma(S) = \begin{cases} S & \text{if } |S| \leq k' \\ \{x_{S_n}\}_{n=1}^{k} & \text{if } |S| > k. \end{cases}$$

Clearly, Γ is idempotent. Consequently, (Γ, u) is a consistent pair of AR-IA, and $u \in \mathcal{U}_{\succcurlyeq}$. k-Cutoff is obviously true. For the converse, when $N = \emptyset$, it is obvious. When $N \neq \emptyset$, we know that there is a u that solves the system of linear inequalities induced by $\succcurlyeq$ on $\bar{\mathcal{B}}$ such that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. By k-Cutoff, for any S there is a $S' \in \bar{B}$ such that $S \sim S'$ and $|S'| = k$. As a result, u also solves the system of inequalities induced by $\succcurlyeq$ on $\{|S| \leq k : S \in \mathcal{X}\}$. □

Γ can be used as a black box when the selection procedure is not as clear as the top k criteria. A consumer chooses between online streaming services and plans to watch TV shows and films for ten hours each month. The amount of time for every $\Gamma(S)$ should not exceed ten hours.

5.2 Freedom of Choice

DMs should prefer a larger menu over a smaller one when their preferences admit AR-IAs. One interpretation of this situation, as argued in Sen (1988), is that DMs are inclined to prefer flexibility or freedom of choice, which indirect utility ranking neglects. According to indirect utility ranking, the provision of an opportunity set is equivalent to providing the most preferred item within it from a utilitarian perspective. Nevertheless, this may not always be the case. Sen (1988) explains that the situation where DMs choose to fast by starving is different from the situation where they are involuntarily starving, although the results are the same.

Sen (1988) introduces the instrumental and intrinsic importance of freedom. The instrumental value of freedom leads to other ends. The DM, for example, prefers to have more choices since the variety of choices will help the DM to mitigate future risks. A suitable example would be the preference for flexibility in Kreps (1979). Instead, freedom's intrinsic value should only be dependent on the preference for freedom itself.

In Pattanaik and Xu (1990), a simple cardinality-based ordering (SCO) is introduced, in which a set S has a greater degree of freedom than a set T if $|S| \geq |T|$. Given the DM's preference $\succcurlyeq$ as the freedom provided in each set, they characterize the SCO with three axioms. One of the axioms is known as Indifference between No-choice Situation (INS) [10].

Axiom 5. *(INS: Indifference between No-choice Situation). For all $x, y \in X$, $\{x\} \sim \{y\}$.*

We consider the SCO under the idempotent attention.

Definition. *$\succcurlyeq$ on $\mathcal{X}$ is an **SCO under idempotent attention (SCO-IA)** if there is an idempotent attention rule $\Gamma : \mathcal{X} \to \mathcal{X}$ such that for any $S, T \in \mathcal{X}$, $S \succcurlyeq T$ if and only if $|\Gamma(S)| \geq |\Gamma(T)|$.*

[10] The other two axioms are (1) Strict Monotonicity (SM): For all distinct $x, y \in X$, $xy \succ x$; and (2) Independence (IND): For all $S, T \in \mathcal{X}$, and $x \in X \setminus (S \cup T)$, $S \succcurlyeq T$ if and only if $S \cup \{x\} \succcurlyeq T \cup \{x\}$.

The mapping Γ in the SCO-IA can be understood as lacking attention. When considering the intrinsic value of freedom, the DM only considers the alternatives in $\Gamma(S)$ for all $S \in \mathcal{X}$. As a result of this interpretation, limited attention has the same meaning as in the AR-IA. Alternatively, limited attention may cause the DM to overlook some states when it comes to the instrumental value of freedom. When the elements of X correspond to states, omitting some items from a menu is equivalent to omitting some states given this menu[11].

Theorem 10. *$\succcurlyeq$ on $\mathcal{X}$ is an SCO-IA if and only if $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA and satisfies INS.*

Proof. Assume that $\succcurlyeq$ on $\mathcal{X}$ is an SCO. By the definition of idempotent attention rules, $\Gamma(x) = x$ for all $x \in X$. For any $x, y \in X$, $|\Gamma(x)| = |x| = |y| = |\Gamma(y)|$ which implies that $x \sim y$. We can let $u : X \to \mathbb{R}^+$ be a constant function such that $u(x) = 1$ for all $x \in X$. Take any $S, T \in \mathcal{X}$, we then have that $S \succcurlyeq T \iff |\Gamma(S)| \geq |\Gamma(T)| \iff \sum_{s\in\Gamma(S)} u(s) \geq \sum_{t\in\Gamma(T)} u(t)$. For the converse direction, assume that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA and $\succcurlyeq$ satisfies INS. We first know that there is a pair of idempotent attention rule Γ and a utility function u where $u(x) = 1$ for all $x \in X$. Hence, $S \succcurlyeq T \iff \sum_{s\in\Gamma(S)} u(s) \geq \sum_{t\in\Gamma(T)} u(t) \iff |\Gamma(S)| \geq |\Gamma(T)|$. □

As long as $\succcurlyeq$ on $\mathcal{X}$ satisfies INS, all alternatives are considered to provide the same degree of freedom by the DM. In the absence of the assumption that alternatives carry the same weight, evaluating freedom of choice is no longer equivalent to comparing cardinalities. Klemisch-Ahlert (1993) investigates this case under the assumption of full attention. This version of measuring freedom of choice will be identical to the AR-IA when the DM's attestation rule is idempotent.

[11] The DM's limited attention on states is different from unawareness. When a collection of plans is presented, the DM may be aware of the existence of some states. However, the DM may overlook them when the collection of plans is altered. We may call this situation limited attention rather than unawareness.

6 Concluding Remarks

This paper extends our understanding of additive representation from the full attention cases to the idempotent attention cases. By focusing on the idempotent attention rules, we provide a characterization of the AR-IA. In addition, AR-AFs are examined as special cases.

The idempotent attention rule is a generalization of attention filter (Masatlioglu et al. (2012)), competition filter (Lleras et al. (2017)), and path independent consideration (Lleras et al. (2021)). However, not all the attention rules used in past papers are idempotent. For example, Geng (2022) introduces the weak competition filter where for any $|S| > 1$, if $x \in \Gamma(S)$, there is a T with $T \subset S$ and $|T| = |S| - 1$ such that $x \in \Gamma(T)$. An idempotent attention rule is not necessarily a weak competition filter.

An advantage of idempotent attention rules is that they separate the attention and aggregation procedures of the DM. Both procedures relate to a subcollection of $\mathcal{X}$, the collection of basic sets. In general, $\mathcal{B}$ can be understood as the image of attention rules when $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Furthermore, the collection of consistent utility functions is determined by $\succcurlyeq$ on $\mathcal{B}$.

Appendix

A Proof of Theorem 1

For the only if part, suppose that $\succsim$ on $\mathcal{X}$ admits the AR-IA under (Γ, u). Let $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m$ be two collections of basic sets such that the DM prefers S_n over T_n for all $n \leq m$, and strictly prefers S_n to T_n for some $n \leq m$. The AR-IA implies that $\sum_{n=1}^m \sum_{s \in T_n} u(s) > \sum_{n=1}^m \sum_{t \in T_n} u(t)$ because basic sets catch full attention. However, if $\sum_{n=1}^m \mathbb{1}_{S_n}(x) \leq \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \in X$, i.e., the collection of S_n contains less x than the collection of T_n for all $x \in X$, then $\sum_{n=1}^m \sum_{s \in T_n} u(s) < \sum_{n=1}^m \sum_{t \in T_n} u(t)$. Therefore, NR is necessary for the AR-IA.

We now move on to the sufficiency of WO and NR.

Step 1. Constructing the utility function by solving the system of linear inequalities induced by the $\succsim$ on $\mathcal{B}$. Let $\mathcal{B} = \{B_n\}_{n=1}^m$. If $B \sim B'$ for all $B, B' \in \mathcal{B}$, then it is obvious that the system of linear inequalities has a solution. We then can assume that there exists at least one pair of $B_n, B_{n'} \in \mathcal{B}$ such that $B_n \succ B_{n'}$. For any $B_n \succ B_{n'}$, we have $\sum_{x \in B_n} u(x) - \sum_{x \in B_{n'}} u(x) > 0$. Similarly, $B_n \sim B_{n'}$ suggests that $\sum_{x \in B_n} u(x) - \sum_{x \in B_{n'}} u(x) = 0$, and $B_n \succsim B_{n'}$ suggests that $\sum_{x \in B_n} u(x) - \sum_{x \in B_{n'}} u(x) \geq 0$. Moreover, we also need a restriction on u, i.e., $u(x) \geq 0$ for all $x \in X$. We then consider the solution to the above system of linear inequalities.

By Kraft et al. (1959), the above system of linear inequalities has a solution if and only if there are no $m_1 > 0$ pairs of $B_n \succ B_{n'}$, $m_2 \geq 0$ pairs of $B_n \sim B_{n'}$, and $m_3 \geq 0$ pairs of $B_n \succsim B_{n'}$ such that the collection of B_n has less x than the collection of $B_{n'}$ for all $x \in X$. Hence, the system of linear inequalities has a solution if and only if $\succsim$ on $\mathcal{B}$ is a weak order and satisfies nonnegative remainders.

Step 2. Constructing Γ by using basic sets. Let $\mathcal{ES}(B_1) = \{S : B_1 \subseteq S \textit{ and } B_1 \sim S\}$ and $\mathcal{ES}(B_i) = \{S : B_i \subseteq S,\ B_i \sim S,\ \textit{and } S \notin \mathcal{ES}_{B_{i-1}}\}$ for all $i > 1$. For every $S \in \mathcal{ES}(B_i)$, let $\Gamma(S) = B_i$. It's clear that Γ is idempotent attention.

Take any pair of (Γ, u) constructed above. For any $S, T \in \mathcal{X}$ with $S \succsim T$, we know that there is a pair of basic sets B_n and $B_{n'}$ such that $S \in \mathcal{ES}(B_n)$ and $T \in \mathcal{ES}(B_m)$.

Therefore,

$$S \succcurlyeq T \iff B_n \succcurlyeq B_{n'} \iff \sum_{x \in B_n} u(x) \geq \sum_{y \in B_{n'}} u(y) \iff \sum_{x \in \Gamma(S)} u(x) \geq \sum_{y \in \Gamma(T)} u(y).$$

B Proof of Proposition 2

Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Let $\{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ be the collection of corresponding idempotent attentions and utility functions.

We first prove the only if part of this proposition. By contradiction, assume that there is a positive integer m and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \succcurlyeq T_n$ for all n, and $S_n \succ T_n$ for some n, such that $\sum_{n=1}^m \mathbb{1}_{S_n}(x) = \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \notin MIN(X, \succcurlyeq)$, and $\sum_{x \in MIN(X,\succcurlyeq)} \sum_{n=1}^m \mathbb{1}_{S_n}(x) > \sum_{x \in MIN(X,\succcurlyeq)} \sum_{n=1}^m \mathbb{1}_{T_n}(x)$. Then,

$$\begin{aligned}
\sum_{n=1}^m \sum_{s \in S_n} u(s) &= \sum_{n=1}^m \sum_{s \in S_n \cap MIN(X,\succcurlyeq)} u(s) + \sum_{n=1}^m \sum_{s \in S_n \setminus MIN(X,\succcurlyeq)} u(s) \\
&> \sum_{n=1}^m \sum_{t \in T_n} u(t) \\
&= \sum_{n=1}^m \sum_{t \in T_n \cap MIN(X,\succcurlyeq)} u(t) + \sum_{n=1}^m \sum_{t \in T_n \setminus MIN(X,\succcurlyeq)} u(t).
\end{aligned}$$

Therefore,$\sum_{n=1}^m \sum_{s \in S_n \cap MIN(X,\succcurlyeq)} u(s) > \sum_{n=1}^m \sum_{t \in T_n \cap MIN(X,\succcurlyeq)} u(t)$. We then have $u(x) > 0$ for all $x \in MIN(X, \succcurlyeq)$.

For the converse direction, suppose that there is a $u \in \mathcal{U}_{\succcurlyeq}$ such that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. We then consider the system of linear inequalities we introduced in the proof of Theorem 1 combining with the restriction that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. By Kraft et al. (1959), this linear system has a solution if and only if there are no $m_1 > 0$ pairs of $B_n \succ B_{n'}$, $m_2 \geq 0$ pairs of $B_n \sim B_{n'}$, $m_3 \geq 0$ pairs of $B_n \succcurlyeq B_{n'}$, and $|MIN(X, \succcurlyeq)| > 0$ restrictions on $u(x) = 0$ where $x \in MIN(X, \succcurlyeq)$ such that the collection of B_n only contains more dominated alternatives than the collection of $B_{n'}$. Hence, we have ENS.

C Proof of Theorem 2

Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-AF under (Γ, u). We first know that (Γ, u) is also a pair of AR-IAs for $\succcurlyeq$ on $\mathcal{X}$. Therefore, we know that $\succcurlyeq$ on $\mathcal{X}$ satisfies WO and NR by Theorem 1.

We then only need to prove that $\succcurlyeq$ on $\mathcal{X}$ satisfies IB. Take any $S \in \mathcal{X}$, we can get $\Gamma(S)$ from the attention filter Γ. We first know that $\Gamma(Y) = \Gamma(S)$ for all $\Gamma(S) \subseteq Y \subseteq S$. If $\Gamma(S)$ is basic, it is clear that $\Gamma(S)$ itself can serve as the corresponding T and B in IB, and $\Gamma(S) \setminus \Gamma(S) = \emptyset \subseteq N$. Moreover, $\sum_{s\in\Gamma(S)} u(s) = \sum_{s\in\Gamma(\Gamma(S))} u(s)$ which implies that $S \sim \Gamma(S)$. We then have IB.

If $\Gamma(S)$ is non-basic, we know that there is a corresponding basic set B of $\Gamma(S)$ such that $B \sim \Gamma(S)$. Hence, $\sum_{s\in\Gamma(S)\setminus B} u(s) = 0$. As a result, $u(s) = 0$ for all $s \in \Gamma(S) \setminus B$. By Proposition 2, we have $\Gamma(S) \setminus B \subseteq N$. We can let $\Gamma(S)$ be the corresponding T of S in IB, and the $\succcurlyeq$ on $\mathcal{X}$ satisfies IB.

We now suppose that $\succcurlyeq$ on $\mathcal{X}$ satisfies WO, NR and IB. We know that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs by Theorem 1. We then need to consider the construction of an attention filer. Let $\{T_i\}_{i=1}^m = \bigcup_{s\in\mathcal{X}} \mathcal{IB}(S)$, and

$$\mathcal{EB}(T,S) := \begin{cases} \{Y : T \subseteq Y \subseteq S\} & \text{if all } Y \text{ inbetween } T \text{ and } S \text{ are indifferent,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Let's also define

$$\mathcal{UEB}(T_1) := \{S : S \in \mathcal{X} \textit{ such that } \emptyset \neq \mathcal{EB}(T_1, S)\},$$

and for all $i > 1$, we denote

$$\mathcal{UEB}(T_i) := \left\{ S : S \in \mathcal{X} \setminus \bigcup_{j<i} \mathcal{UEB}(A_j) \textit{ such that } \emptyset \neq \mathcal{EB}(T_i, S) \right\}.$$

Claim 1. *$\{\mathcal{UEB}(T_i)\}_{i=1}^m$ forms a partition of $\mathcal{X}$.*

Proof. We first show that for any set S, there is a $i \leq m$ such that $S \in \mathcal{UEB}(T_i)$.

Take any $S \in \mathcal{X}$, we know that there exists at least one of T_i such that $T_i \in \mathcal{IB}(S)$. We then can denote the $\mathcal{IB}(S)$ as a finite subsequence $\{T_{i_j}\}_{j=1}^n$ of $\{T_i\}_{i=1}^m$. We then claim that $S \in \mathcal{UEB}(T_{i_1})$. To show this, we only need to show that $S \notin \mathcal{UEB}(T_i)$ where $i < i_1$. By contradiction, suppose that $S \in \mathcal{UEB}(T_i)$ for some $i < i_1$. Then, we know that $T_i \in \mathcal{IB}(S)$ which is a contradiction.

We then show that for any $T_i, T_j \in \{T_i\}_{i=1}^m$ where $i \neq j$, $\mathcal{UEB}(T_i) \cap \mathcal{UEB}(T_j) = \emptyset$. Without loss of generality, we can assume that $i > j$. By contradiction, suppose that there is a set S such that $S \in \mathcal{UEB}(T_i) \cap \mathcal{UEB}(T_j)$, we then know that $S \in \mathcal{EB}(T_j, S)$ which implies that $S \notin \mathcal{X} \setminus \bigcup_{j<i} \mathcal{UEB}(T_j)$. Hence, $S \notin \mathcal{UEB}(T_i)$.

In conclusion, $\{\mathcal{UEB}(T_i)\}_{i=1}^m$ forms a partition of $\mathcal{X}$. □

Let's define a function $\Gamma : \mathcal{X} \to \mathcal{X}$ by $\Gamma(S) = T_i$ if $S \in \mathcal{UEB}(T_i)$. We then claim that Γ is an attention filter. Take any $S \in \mathcal{X}$, and suppose that $\Gamma(S) = T_i$. Take any Y with $T_i \subseteq Y \subseteq S$, we want to show that $\Gamma(Y) = T_i$. We first know that $Y \sim S$. We then only need to show that $Y \notin \mathcal{UEB}(T_j)$ for any $j < i$. By contradiction, if $Y \in \mathcal{UEB}(T_j)$ for some $j < i$, then we know that $S \in \mathcal{UEB}(T_j)$. It is a contradiction. Therefore, Γ is an attention filter.

We now want to show that there is a utility function u such that $S \succcurlyeq T \iff \sum_{s\in S} u(s) \geq \sum_{t\in T} u(t)$. Take any $S \succcurlyeq Y$, we know that there are two basic sets T_S and T_Y such that $\Gamma(S) = T_S$ and $\Gamma(Y) = T_Y$. By IB, there is a basic set B_1 where $B_1 \subseteq T_S$ and $B_1 \sim T_S$ such that $T_S \setminus B_1 \subset N$. Similarly, we can get a basic set B_2 for T_Y. For all u that consist of AR-IAs, we have

$$\begin{aligned} S \succcurlyeq Y &\iff T_S \succcurlyeq T_Y \\ &\iff B_1 \succcurlyeq B_2 \\ &\iff \sum_{b\in B_1} u(b) \geq \sum_{b\in B_2} u(b) \\ &\iff \sum_{s\in\Gamma(S)} u(s) \geq \sum_{y\in\Gamma(Y)} u(y). \end{aligned}$$

D Independence between WO, NR and IB

From Example 2, we can see that there is a preference relation $\succcurlyeq$ on $\mathcal{X}$ that admits AR-IAs but not AR-AFs. To see the $\succcurlyeq$ provided in Example 2 violates IB, let us consider xyz. First, we know that $N = \emptyset$. xyz is not basic suggests that $\mathcal{IB}(xyz) = \emptyset$.

We now want to show that WO and IB together cannot characterize AR-AFs.

Example 6. *Let $X = xy$, and the $\succcurlyeq$ on $\mathcal{X}$ is $x \succ xy \succ y$. We know that $N = \emptyset$, and all the sets are basic. Then it satisfies WO and IB. However, it does not admit AR-AFs because $x \succ xy \succ y \iff u(y) < 0$.*

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